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A STRAIGHTFORWARD GENERAL ANALYSIS OF SIGNAL DISTORTION
WITH APPLICATIONS TO WIDEBAND IONOSPHERIC DISPERSION

1 of 2

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MARCH 1968

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Prepared for

DEVELOPMENT ENGINEERING DIVISION
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

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FOREWORD

The work reported in this document was performed by The MITRE Corporation, Bedford, Massachusetts, for the Development Engineering Division, Electronic Systems Division, of the Air Force Systems Command under Contract AF 19(628)-5165.

REVIEW AND APPROVAL

Publication of this technical report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

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Development Engineering Division
Directorate of Planning and Technology

ABSTRACT

The degradation of signal processor performance in terms of loss of resolution (i.e., increased main lobe width and increased sidelobes) is analyzed and calculated for dispersive media. The analysis gives insight into the distortion mechanisms and compares well with straightforward and precise numerical calculations. The distortion of wideband signals is described for the following situations: Ionospheric dispersion (above plasma frequency), waveguide operated near cutoff, time dilation, and frequency dependent scattering cross section.

ACKNOWLEDGMENT

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SECTION I

SUMMARY OF DISTORTION ANALYSIS

In many communications and radar systems the signals are
distorted by reflection from moving targets or transmission through
dispersive media such as the ionosphere or waveguide. A general
analysis of these effects is made and compared to straightforward
but general numerical calculations which evaluate the distortion.
The analysis and calculations are made for signals with a flat-
amplitude band-limited spectrum such as linear FM pulse or an ideal
bandpass impulse. Quadratic phase distortion is given the most
emphasis because it is produced by several common phenomena such as
ionosphere dispersion, waveguide dispersion, and time dilation
distortion.

The effects of the distortion are determined in terms of radar
performance. This includes both detection capability (signal-to-
noise ratio), resolution capability (ability to distinguish nearby
targets—main lobe width, and ability to distinguish targets of
greatly different cross section—sidelobe level). In a communications
context parameters such as intersymbol interference and probability of
error would be important. Although these parameters are not discussed,
it would be possible to easily relate them to the analysis presented.

The purpose of this paper is to understand distortion in detail,
to develop a technique applicable to a broad class of distortion

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mechanisms, and to determine the loss in signal-to-noise ratio, loss in range resolution (i.e., the loss in main lobe width and attainable sidelobe level) that result from common problems such as ionosphere dispersion, waveguide dispersion, and time dilation.

A general form of paired echo theory is used to relate the phase distortion to the waveform at the output of a filter matched to the uncorrupted signal. Because the signal is assumed to be band-limited, it may be represented by a set of samples in the time domain. For small phase errors these samples are simply related to the Fourier coefficients of the phase error function.

Phase functions which have a strong periodic component generally introduce a large sidelobe. Phase functions which are slowly varying and can therefore be represented by a few terms of a power series are more difficult to analyze. The Fourier coefficients predict that there will be a considerable amount of sidelobes due to the phase distortion. The Fourier coefficients are related in amplitude and phase in such a way that sidelobe weighting (e.g., Taylor weighting) significantly reduces the distortion sidelobes. Therefore, the major effect of this type of phase function is to decrease the signal resolution by broadening the main lobe rather than by increasing the sidelobes.

BACKGROUND

The mathematics used to describe the ambiguity function for time signals is very similar to the mathematics for computing antenna

pattern. The concept of sidelobe weighting frequently used in signal processing context was originally conceived to improve antenna patterns.[1] Because antenna development has led the development of sophisticated signal processing techniques, many of the papers on the effects of phase distortion on ambiguity functions (antenna pattern) can be found in the literature on antenna theory.[2,3,4]

More recently, papers on the processing of time signals have appeared. An early paper by Klauder et al[5] gives a comprehensive treatment of linear FM signal processing and includes a discussion of quadratic phase distortion. A paper by Elliott "Pulse Waveform Degradation Due to Dispersion in Waveguide"[6] treats quadratic phase distortion of sinusoidal pulses (uncoded). Like Klauder, Elliott uses Fresnel integrals to evaluate the nonlinear phase effects.

A paper by Brookner, "Effect of Ionosphere on Radar Waveforms",[7] which relates the frequency dependent phase velocity of an electromagnetic wave propagating through the ionosphere to a quadratic phase function, uses the results of Klauder and Elliott to define the effective bandwidth of the ionosphere.

For many applications, such as high performance, long range radar a more complete understanding of the distortion process and a more precise means of specifying the effect of the distortion is required. In some applications it may be necessary to achieve

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sidelobes as low as -55 db relative to the main lobe, while simultaneously maintaining narrow main lobe and high signal-to-noise ratio.

SECTION II

GENERAL FOURIER DISTORTION ANALYSIS

In this section the degradation of a signal transmitted through
a linear filter is found simply and without error by Fourier trans-
forming the product of the frequency characteristic of the distorting
filter and the squared magnitude of the signal characteristic. This
relation has been recently pointed out by R.D. Haggarty who
analyzed the effects of errors in the hardware implementation of a
large time-bandwidth filter. This procedure is a more general
calculation of the type often referred to as "paired echoes". [5]
The effects of sidelobe weighting are easily included by suitably
modifying the filter characteristic.

Consider the block diagram in Figure 1, where the following
filters are represented:

1. A linear time-invariant filter with impulse
   response $x(t)$, and frequency characteristic
   $X(f)$. The filter output represents the
   transmitted waveform.

2. A linear time-invariant filter with frequency
   characteristic, $I(f)$. This filter represents
   the distorting media.

3. A filter matched to the transmitted signal. Its
   frequency characteristic is therefore $H(f) = X^*(f)$,

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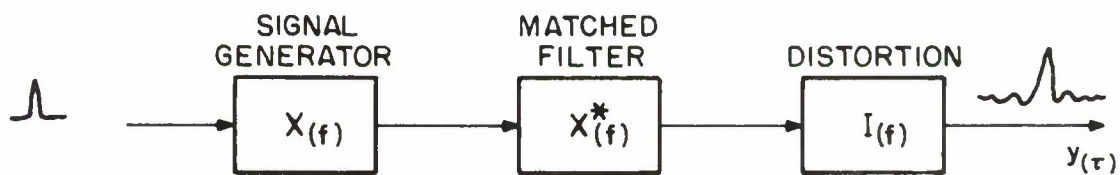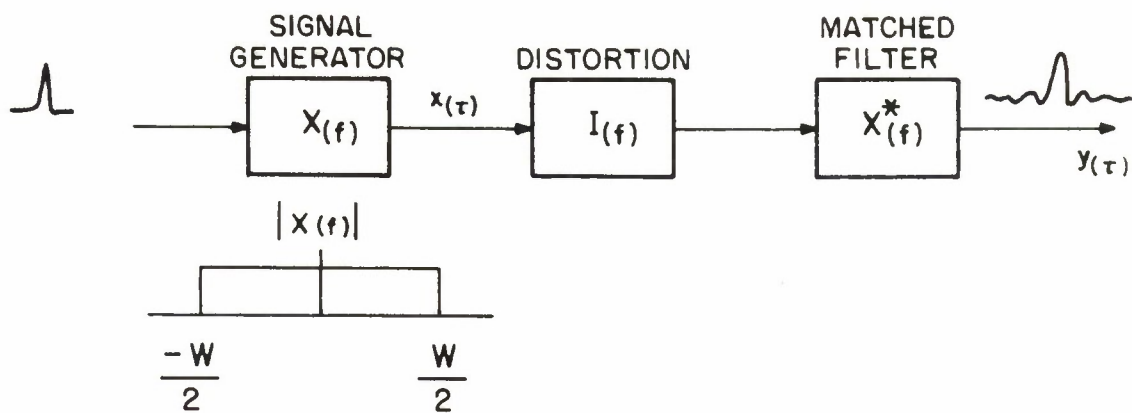
Figure 1. Linear Filter Representation

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where the symbol $*$ denotes a complex conjugate.

Figure 1 also shows the same system with the matched filter and the distorting filter interchanged. The output of the system, $y(t)$ (namely the received signal) is not affected by the interchange because linear time-invariant filter characteristics are commutative (i.e., $I(f) X^*(f) = X^*(f) I(f)$).

The observation is made that the phase of $X(f)$ does not affect the output (received) signal $y(t)$. Defining the transform of $y(t)$ as $Y(f)$ the following relationship is shown:

$$y(t) = \int_{f_o - \frac{W}{2}}^{f_o + \frac{W}{2}} Y(f) \; e^{j2\pi ft} \; df$$

$$= \int_{f_o - \frac{W}{2}}^{f_o + \frac{W}{2}} X(f) \; I(f) \; H(f) \; e^{j2\pi ft} \; df$$

$$y(t) = \int_{f_o - \frac{W}{2}}^{f_o + \frac{W}{2}} |X(f)|^2 \; I(f) \; e^{j2\pi ft} \; df \tag{1}$$

where f_o is the center frequency

and W is the bandwidth.

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The output of a matched filter is often modified to reduce its sidelobes by introducing another filter with response $W(f)$. A simple form of sidelobe weighting uses the function $W(f) = 1 + \cos\frac{2\pi f}{W}$. The output of the system with weighting, $W(f)$, is given by

$$z(t) = \int_{f_o - \frac{W}{2}}^{f_o + \frac{W}{2}} |X(f)|^2 \; I(f) \; W(f) \; e^{j2\pi ft} \; df \qquad (2)$$

If $x(t)$ is a complicated waveform (e.g., linear FM) a considerable amount of calculation time can be saved by using this expression. For a large time bandwidth linear FM signal* with a rectangular time envelope the following relationship approximately holds

$$|X(f)|^2 = \begin{cases} 1 & f_o - \frac{W}{2} \leq f \leq f_o + \frac{W}{2} \\ \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

Therefore instead of evaluating complicated Fresnel integrals to determine $y(t)$ a simple integral needs to be evaluated

$$z(t) = \int_{f_o - \frac{W}{2}}^{f_o + \frac{W}{2}} I(f) \; W(f) \; e^{j2\pi ft} \; df \qquad (4)$$

---

*Note that for the commonly realized linearly dispersive filter systems, Equation 3 is the ideal (error free) result.

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An understanding of distortion effects can be achieved by appealing to the sampling theorem or to a general paired echo theory. Paired echo theory predicts [1] that if a signal $\psi(t)$ is applied to a filter with the amplitude and phase spectra

$$A(f) = 1$$
$$\phi(f) = b_1 \sin 2\pi fc$$
(5)

that the output, $Z(t)$, consists of a series of signals differentially delayed.

$$Z(t) = \psi(t) + J_1(b_1)[\psi(t + c) - \psi(t - c)] + J_2(b_1)[\psi(t +2c) + \psi(t - 2c)]$$

$$+ J_3(b_1)[\psi(t + 3c) - \psi(t - 3c)] + + +$$
(6)

The Bessel function, $J_2(b_1)$, can be approximated for small values of their argument,

$$J_1(b_1) \approx \frac{b_1}{2}$$

$$J_2(b_1) \approx J_3(b_1) \approx 0 \qquad\qquad \text{if} \quad b_1 \ll 1 \qquad (7)$$

The sampling theorem states that a function limited to a band W may be represented by the following series

$$y(t) = \sum_{k=-\infty}^{+\infty} y_k \frac{\sin \pi W(t - \frac{k}{W})}{\pi W(t - \frac{k}{W})}$$
(8)

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$$= \sum y_k \ \psi_k(t)$$

where $\quad y_k = y(\frac{k}{W})$

$$\psi_k(t) = \frac{\sin \pi W(t - \frac{k}{W})}{\pi W(t - \frac{k}{W})} \qquad (9)$$

Equation 4 can therefore be rewritten as

$$y_k = \int\limits_{f_o - \frac{W}{2}}^{f_o + \frac{W}{2}} I(f) \ e^{\frac{j2\pi fk}{W}} \ df \qquad (10)$$

for the case without sidelobe weighting.

If there is no distortion

$$I(f) = constant = \frac{1}{W} \qquad (11)$$

then

$$y_o = 1$$
$$\qquad\qquad\qquad i \neq 0 \qquad (12)$$
$$y_i = 0$$

On the other hand, if $I(f)$ has a periodic amplitude component

$$I(f) = \frac{1}{W} \left[1 + a_1 \sin 5\left(\frac{2\pi f}{W}\right) \right] \qquad (13)$$

then

$$y_o = 1$$

$$y_5 = + \frac{ja_1}{2}$$

$$y_{-5} = - \frac{ja_1}{2} \qquad (14)$$

$$y_i = 0 \qquad\qquad i \neq 0, 5, -5$$

For small periodic phase errors the problem reduces to this case. Let

$$|I(f)| = \frac{1}{W}$$

$$\angle I(f) = \theta(f) = b_1 \sin \frac{5 \cdot 2\pi f}{W} \qquad (15)$$

then for $b_1 \ll 1$ a Taylor series expansion of the exponential gives

$$I(f) = \frac{1}{W} e^{j\theta(f)} \approx \frac{1}{W} \left[1 + j\theta(f) \right] \qquad (16)$$

The periodic phase error therefore reduces to the periodic amplitude error case by letting $a_1 = jb_1$. The time function is the same as that predicted by the paired echo theory.

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$$y(t) = -\frac{b_1}{2} \psi(t - \frac{5}{W}) + \psi(t) + \frac{b_1}{2} \psi(t + \frac{5}{W}) \qquad (17)$$

In many practical situations there may be both amplitude and phase sinusoidal modulation. These may be related in such a way that paired echoes do not appear but only a series of echoes following the main lobe appear. Multipath communications links or nonideal termination in cables or delay lines are examples in which this phenomenon is common.

Sidelobe weighting can be easily incorporated into this analysis. Appealing to paired echo theory the output from the periodic phase distorting filter given in Equation 15 can be found directly to be

$$z(t) = -\frac{b_1}{2} \chi(t - \frac{5}{W}) + \chi(t) + \frac{b_1}{2} \chi(t + \frac{5}{W}) \qquad (18)$$

where $\chi(t)$ is the weighted undistorted output signal.

The sampling theorem approach is also straightforward. The undistorted weighted output can be found by multiplication of the frequency characteristic shown in Equation 10 or convolving the transforms of their characteristics. If $w(t)$ is the transform of $W(f)$, the weighting function, then

$$\chi(t) = y(t) \otimes w(t) \qquad (19)$$

where $\otimes$ implies a convolution.

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The distorted output is given by

$$z(t) = \sum_i y_i \ \psi_i(t) \otimes w(t) \tag{20}$$

$$= \sum_i y_i \ \chi_i(t)$$

where $\chi_i(t) = \chi(t - \frac{i}{W})$

which reduces to Equation 18 by substitution for y_i

$$y_o = 1$$

$$y_{+5} = - \frac{b_1}{2}$$

$$y_{-5} = \frac{b_1}{2}$$

$$y_j = 0 \qquad j \neq 0, \ 5, \ -5$$

This discussion can be summarized by writing a general expression for the output time signal as a function of the Fourier series coefficients, λ_i , of the phase distortion.

To summarize, for small phase errors the unweighted and weighted distorted signals are related to the Fourier coefficients of the distortion. Therefore, if

$$\phi(f) = \sum_i \lambda_i \ \sin\left(\frac{2\pi f_i}{W}\right)$$

and

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$$\phi(f) \ll 1 \text{ radian}$$

then the unweighted signal is

$$y(t) = \sum \lambda_i \ \psi(t - \frac{i}{W})$$ (21)

and the weighted signal is

$$z(t) = \sum \lambda_i \ \chi(t - \frac{i}{W})$$ (22)

Note that Equation 21 is an expansion of the function $y(t)$ on an orthogonal basis, $\psi(t)$, and that Equation 22 is an expansion of the function, $z(t)$, on a non-orthogonal basis, $\chi(t)$. Because $z(t)$ is a band-limited function, it can also be represented on an orthogonal basis also,

$$z(t) = \sum z_i \ \psi(t - \frac{i}{W})$$ (23)

The coefficients, z_i , are not trivially related to the Fourier series coefficients, λ_i of the distortion, as is discussed in a latter section. The advantage of the non-orthogonal expansion (Equation 22) is that the coefficients are the Fourier series coefficients, λ_i , of the distortion function. Because the $\psi(t)$ are $\frac{\sin x}{x}$ functions the unweighted output at some time $t = \frac{k}{W}$ is

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$$y(\frac{k}{W}) = \lambda_k \qquad\qquad (24)$$

The ψ functions form an orthogonal set and in the unweighted case distortion sidelobes are simply related to the Fourier coefficients λ_i . The weighted output does not have this property. In general

$$z(\frac{k}{W}) = \sum_k \lambda_i \; \chi \left[\frac{(k - i)}{W} \right] \qquad\qquad (25)$$

The χ functions are not orthogonal and therefore the actual sidelobes are a complicated function of the Fourier coefficients, λ_i .

To illustrate the importance of paired-echo theory the integral specifying the output signal (Equation 4) is numerically evaluated for a periodic phase distortion (Equation 13). The calculation made without weighting and with 60 db Taylor weighting is shown in Figure 2.

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$$\phi_\epsilon = 0.1 \ \text{SIN} \left[\frac{5 \ (2\pi f)}{W} \right]$$

$$\text{MAX} \ \phi_\epsilon = .1 \ \text{RAD.} = 6°$$

NO WEIGHTING

60 db WEIGHTING

26 db

IB—22,326

y (t)

t/W
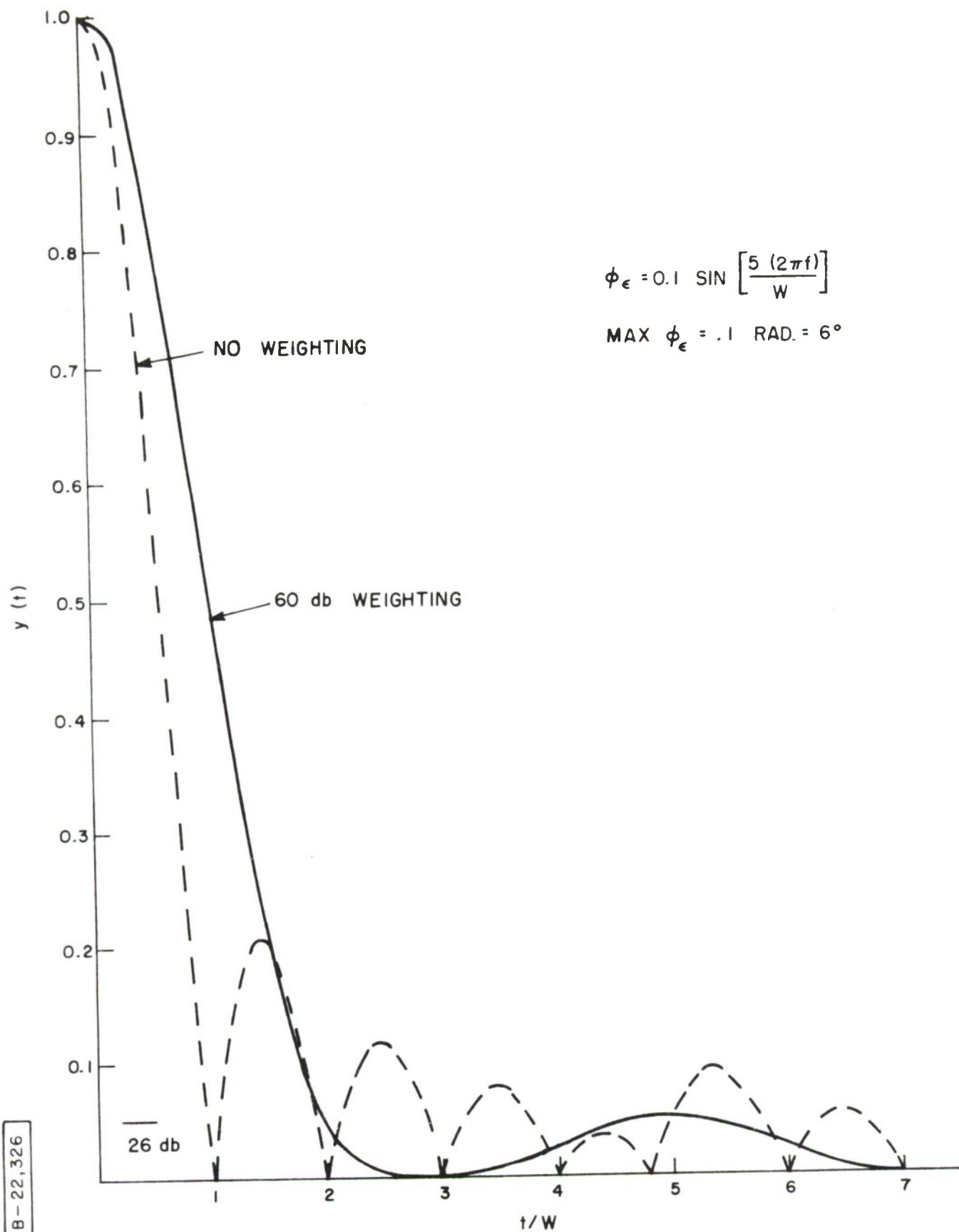
Figure 2. Output Time Signal Sinusoidal Phase Error

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SECTION III

QUADRATIC PHASE DISTORTION

This section treats the analysis of the effect of small
quadratic phase distortion. A linear phase function delays a
signal but does not introduce a distortion. Quadratic phase functions
can distort a signal significantly and are inherent in waveguide
transmission, ionospheric propagation, and time dilation. The analysis
is made with the signal spectrum and phase distortion function defined
in the frequency domain thus the distortion causes range (time)
sidelobes. The dual case of a time signal and time dependent phase
function which introduces doppler (frequency) sidelobes is briefly
discussed.

The quadratic phase function (plotted in Figure 3) can be
expressed conveniently as

$$\theta(f) = \frac{4\Phi}{W^2}(f - f_o)^2 \qquad \text{for} \quad f_o - \frac{W}{2} \leq f \leq f_o + \frac{W}{2} \qquad (26)$$

The maximum phase shift at the edge of the band is

$$\theta_{max} = \theta(\frac{W}{2}) = \Phi \qquad (27)$$

The quadratic phase function (vs frequency) can also be
characterized by the change in group delay across the band. The group
delay is defined

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$$\tau g(f) = -\frac{1}{2\pi}\frac{d\phi(f)}{df}$$

DISPERSIVE DELAY
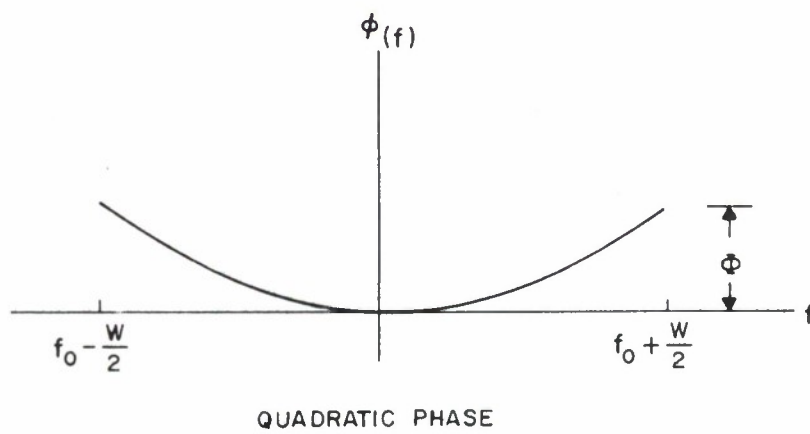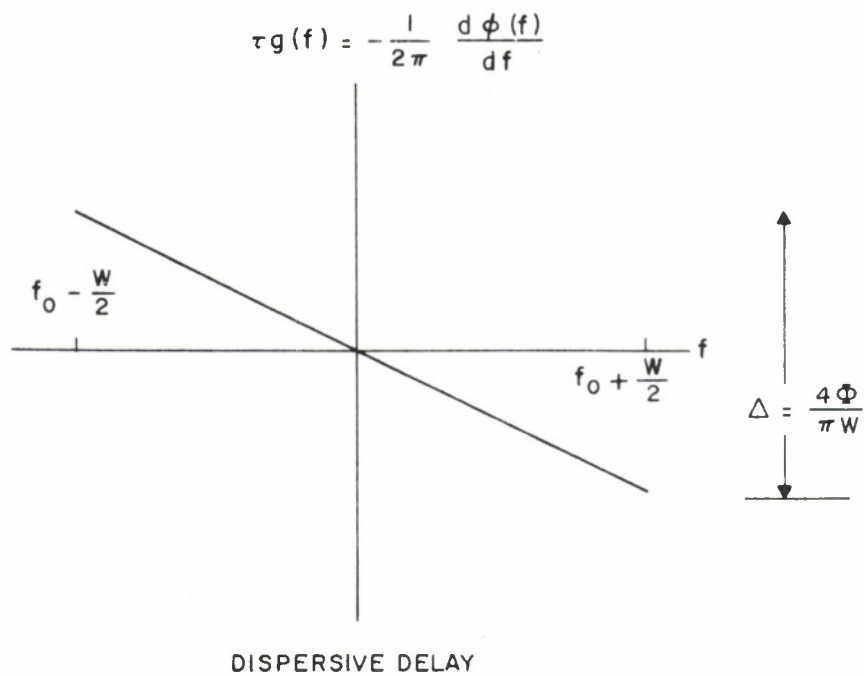
QUADRATIC PHASE

Figure 3. Dispersive Frequency Characteristic

$$\tau_g \triangleq - \frac{1}{2\pi} \frac{d\theta(f)}{df} \tag{28}$$

For quadratic phase the group delay is

$$\tau_g = - \frac{4\Phi(f - f_o)}{\pi W^2} \tag{29}$$

The maximum group delay (see Figure 3) is often called the "dispersion" and is:

$$\tau_{g\ max} = \Delta = \frac{4\Phi}{\pi W} \tag{30}$$

The product of the total change in group delay across the band and bandwidth is defined as the time-bandwidth product and is

$$W\Delta = \frac{4\Phi}{\pi} \tag{31}$$

Using Equation 1 the unweighted distorted signal is found for the case where

$$|I(f)| = const = \frac{1}{W}$$

$$\angle\ I(f) = \theta(f)$$

$$y(t) = \frac{1}{W}\ e^{j2\pi f_o t} \int_{-\frac{W}{2}}^{+\frac{W}{2}} e^{\frac{j4\Phi f^2}{W^2}}\ e^{j2\pi ft}\ df \tag{32}$$

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For the remainder of the paper the center frequency is assumed zero for convenience; no generality is lost because of the use of complex notation.[*] This integral cannot be exactly evaluated in closed form for an arbitrary value of Φ but can only be approximated by using tabulated Fresnel integrals. However, if attention is restricted to small values of Φ an approximate analysis can be made. Therefore, for $\Phi \ll 1$ radian

$$e^{j\theta(f)} \approx \frac{1}{W}\left(1 + j\,\frac{4\Phi f^2}{W^2}\right) \tag{33}$$

The output signal in this case can be expressed as the sum of two signals, one representing the uncorrupted signal, the other the distortion.

$$y(t) = \frac{1}{W}\int_{-\frac{W}{2}}^{+\frac{W}{2}} e^{j2\pi ft}\,df + \int_{-\frac{W}{2}}^{+\frac{W}{2}} \frac{j4\Phi}{W^2}\,f^2\,e^{j2\pi ft}\,df \tag{34}$$

$$y(t) = y_1(t) + y_2(t)$$

The first term is recognized as the transform of a rect function

$$y_1(t) = \frac{\sin \pi Wt}{\pi Wt} = \psi(t) \tag{35}$$

_____

[*]Note that the narrow band assumption is made throughout.

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and represents the desired output of the system if no distortion is
encountered. The second integral is recognized as

$$y_2(t) = -j \frac{4\Phi}{W^2} \frac{1}{(2\pi)^2} \ddot{y}_1(t) \tag{36}$$

where $\ddot{y}_1(t) = \frac{d^2}{dt^2} [y_1(t)]$

and is the result of the phase distortion.

Evaluation of the derivatives yields

$$\ddot{y}_1(t) = -\frac{\pi W}{t} \sin \pi Wt - \frac{2}{t^2} \cos \pi Wt + \frac{2 \sin \pi Wt}{\pi Wt^3} \tag{37}$$

As discussed in the previous section, a convenient way of expressing
this function is to evaluate it at $t = \frac{k}{W}$. The resulting numbers are
the Fourier coefficients of the phase function expansion and also the
paired echoes of the received signal

$$y_2(\frac{k}{W}) = \lambda_k = + \frac{j2\Phi(-1)^k}{\pi^2(k)^2} \tag{38}$$

From this expression several important observations can be made:

  1. The Fourier coefficients are small relative
      to one because the $\frac{1}{\pi^2}$ factor reduces them by
      about 20 db.

  2. The coefficients drop off as the square of $1/k$.

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Thus, the tenth coefficient is 40 db below the

first coefficient.

3. The sign of the coefficients alternate.

4. The phase of the distortion coefficients is

$\pm 90^{\circ}$ relative to the desired zeroth order

coefficient.

5. For $k = 0$ the limit may be taken

$$y_2(\frac{0}{W}) = - \Phi$$

An analysis of the dual situation can be made and is helpful in

understanding the distortion mechanism. For this dual case the

quadratic phase becomes a function of time and causes sidelobes in

the frequency domain. Time dilation and errors common in active

signal generators are distortions of this type.

Let the distorted time signal be

$$r(t) = \frac{1}{T} e^{\frac{j4\Phi t^2}{T^2}} \qquad \text{for } -\frac{T}{2} \le t \le +\frac{T}{2} \qquad (39)$$

The quantity Φ is again the maximum quadratic phase shift, and occurs

at $t = \pm \frac{T}{2}$. In a manner almost identical to the previous analysis

it can be shown that the output of a spectrum analyzer (matched filter)

is given by

$$R(f) = R_1(f) + R_2(f)$$

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$$R(f) = R_1(f) - \frac{j4\Phi}{(2\pi)^2 T} \frac{d^2 R_1(f)}{df^2} \tag{40}$$

where $R_1(f) = \dfrac{\sin \pi T f}{\pi T f}$

The distortion coefficients (paired echoes) are given by

$$R_2\left(\frac{\ell}{T}\right) = + \frac{j2\Phi(-1)^\ell}{\pi^2 \ell^2} \tag{41}$$

The calculations for frequency dependent phase distortion functions were checked by numerical computation. An SDS-930 digital computer was used to evaluate the integral in Equation 1. The calculation is performed for the following values of quadratic phase:

$$\Phi = 0.0, \ 0.4, \ 1.0, \ 5.0, \ \text{and } 20.0 \text{ radians}$$

The results are plotted in Figures 4 and 5. Equation 38 predicted that for $\Phi = 1$ radian the output time signal would be

$$y\left(\frac{1}{W}\right) = .2$$

$$y\left(\frac{2}{W}\right) = .05$$

$$y\left(\frac{3}{W}\right) = .02$$

Figure 4 verifies this result almost exactly. The computer calculation also verified that the phase alternated $\pm 90^\circ$ relative to $y\left(\frac{0}{W}\right)$.

23
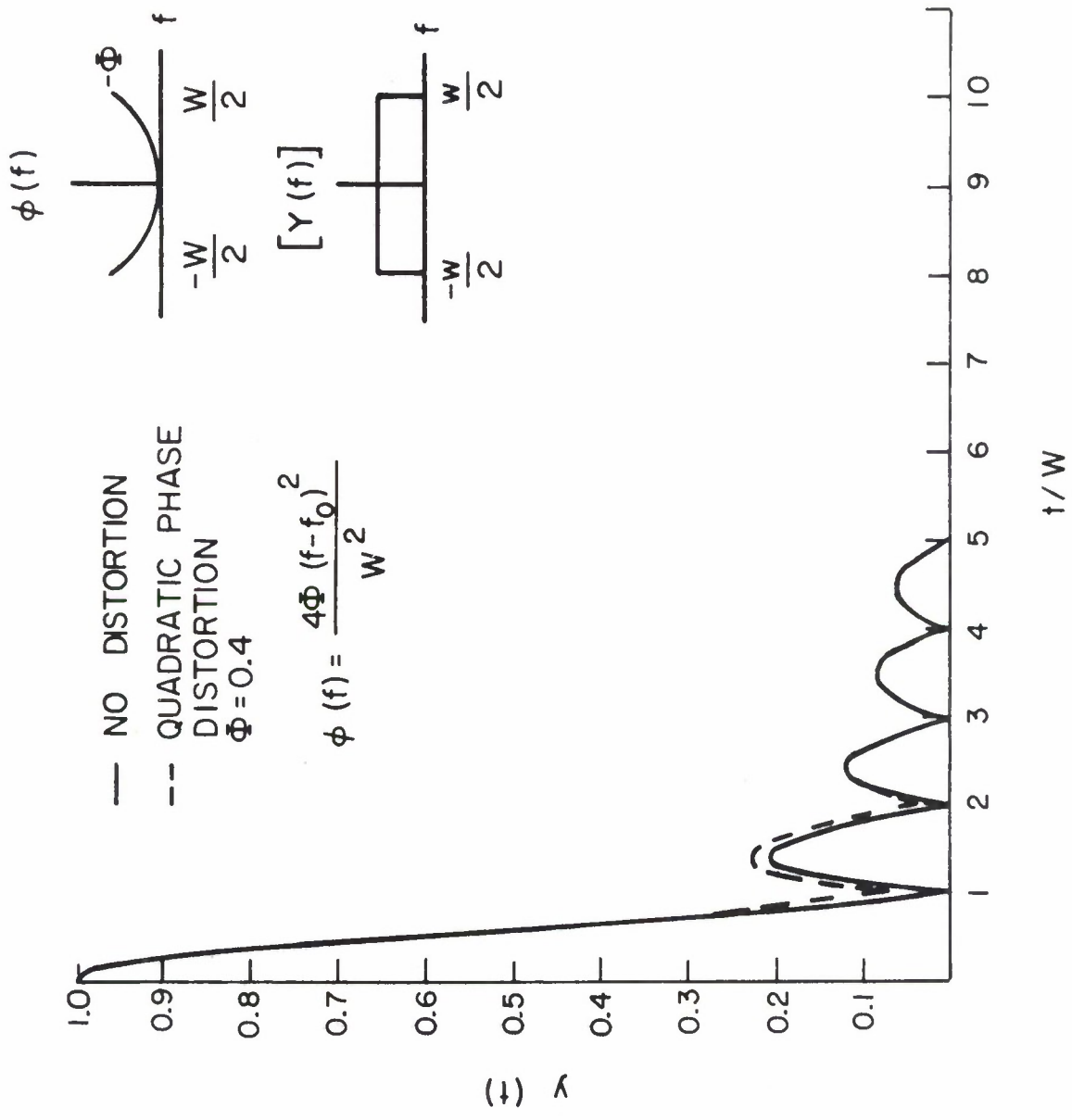
Figure 4. Output Time Signal No Weighting

The figure shows a plot of $y(t)$ versus t/W. The curves show the output time signal for two cases:

- — NO DISTORTION
- -- QUADRATIC PHASE DISTORTION $\Phi = 0.4$

$$\phi(f) = \frac{4\Phi(f-f_0)^2}{W^2}$$

Phase diagram: $\phi(f)$ versus f, showing $-\frac{W}{2}$, $\frac{W}{2}$, and $-\Phi$

Spectrum diagram: $[Y(f)]$ versus f, showing $-\frac{W}{2}$, $\frac{W}{2}$

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IA-22,324
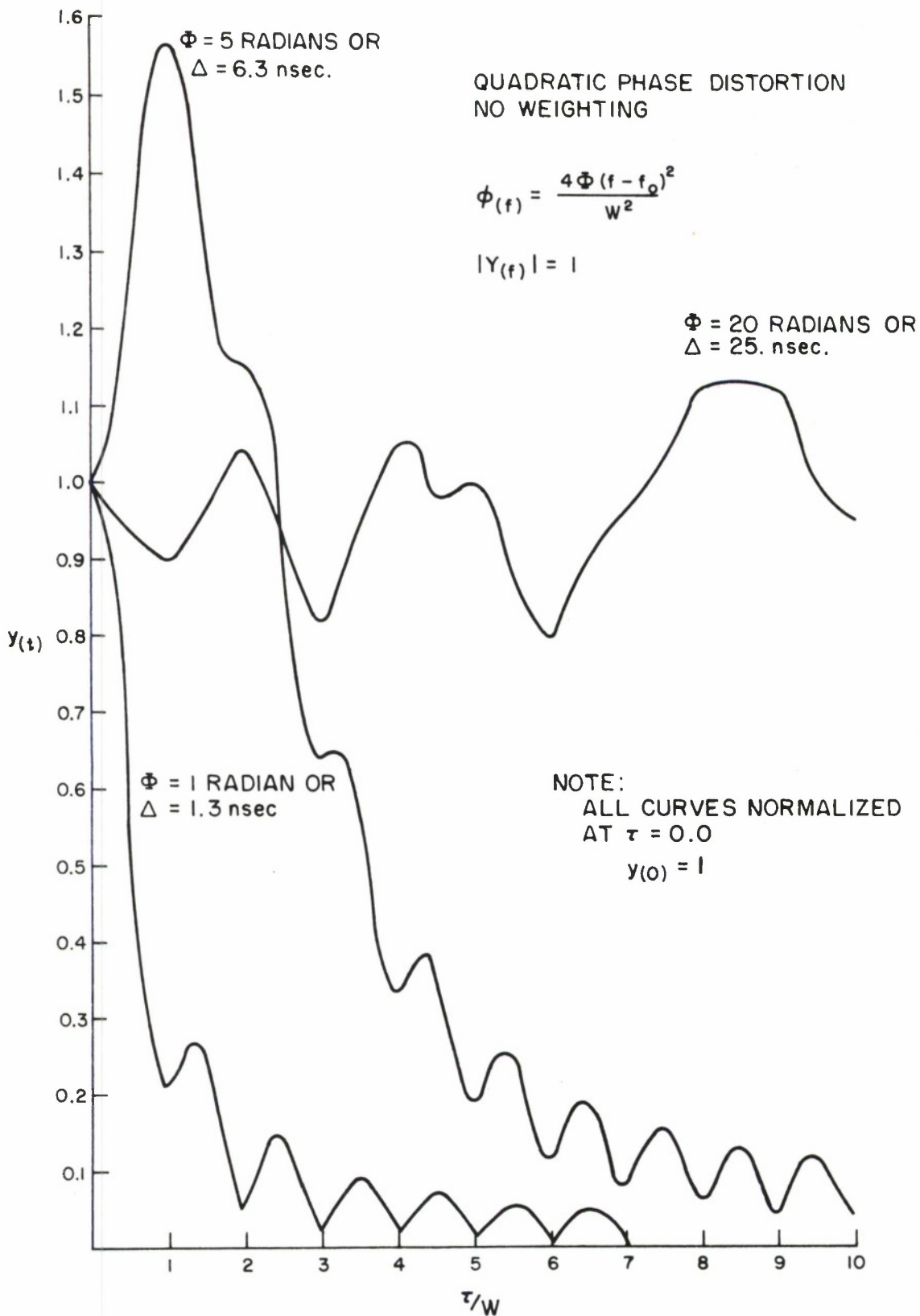
Figure 5. Output Time Signal

SECTION IV

EFFECTS OF SIDELOBE WEIGHTING

The concept of increasing resolution by modifying the spectrum
of a signal to minimize its sidelobe level while simultaneously main-
taining high signal-to-noise ratio and narrow main lobe response has
been well documented. [1,5] Although the Klauder paper [5] does
present some useful data on the effects of weighting on quadratic
phase distortion, there are several questions which remain unanswered.
These include:

1. Is there a simple model to predict the effect
   of weighting for quadratic phase distortion
   or for a wider class of distortion?

2. Is it possible to achieve very low sidelobes
   (-40 to -60 db) for moderate quadratic phase
   distortion?

3. To what extent is the main lobe broadened by
   the combination of distortion and weighting;
   i.e., what is the loss in bandwidth.

THEORETICAL DISCUSSION

The answer to the first question is yes, as is shown in the
following discussion. The weighted undistorted time signal was given
in Equation 19.

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$$\chi(t) = \frac{\sin \pi Wt}{\pi Wt} \otimes w(t)$$

$$+ \frac{W}{2} \tag{42}$$

$$\chi(t) = \int\limits_{-\frac{W}{2}}^{\frac{W}{2}} W(f) \, e^{j2\pi ft} df$$

The distorted output time signal was given in Equation 22

$$Z(t) = \sum_{i=-\infty}^{\infty} \lambda_i \; \chi \left(t - \frac{i}{W} \right) \tag{43}$$

where λ_i are the Fourier series coefficients of the phase distortion function.

But $w(t)$ itself is bandlimited and can be represented by samples

$$w(t) = \sum w_k \; \frac{\sin \pi W \left(t - \frac{k}{W} \right)}{\pi W \left(t - \frac{k}{W} \right)} \tag{44}$$

Therefore, the weighted distorted time signal is given by combining Equation 20 and 44 and performing the convolution

$$z(t) = \sum_{\ell=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} \lambda_k \, w_{\ell-k} \right] \frac{\sin \pi Wt}{\pi Wt}$$

$$= \sum_{\ell=-\infty}^{\infty} Z_i \; \frac{\sin \pi Wt}{\pi Wt} = \sum Z_i \; \Psi_i(t) \tag{45}$$

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This equation is the same as Equation 23 and is the orthogonal expansion of $z(t)$.

The coefficients of the weighting curve, w_k are usually very small for k larger than 10. For example 35 db Taylor weighting can be achieved with a maximum k of 5. For analytical convenience one plus cosine weighting is used (see Figure 6), where

$$W(f) = 1 \pm \cos 2\pi \frac{(f - f_o)}{W} \tag{46}$$

$$w_{-1} = \frac{1}{2} \qquad w_o = 1 \qquad w_1 = \frac{1}{2}$$

Approximately 30 db sidelobes can be achieved using this very simple form of sidelobe weighting. This form of weighting as well as the Taylor weighting discussed later are monotonically decreasing functions. The analysis and the numerical results apply only to this class of weighting function.

Using Equation 45 and 46 the samples of the weighted output signal are

$$z_\ell = \frac{1}{2} \lambda_{\ell-1} + \lambda_\ell + \frac{1}{2} \lambda_{\ell+1} \tag{47}$$

Using the Fourier coefficients for quadratic phase distortion in Equation 38 and this relationship, the expression for the samples of $z(t)$ can be written as the sum of two functions; the first represents the undistorted output, the second the distortion

28
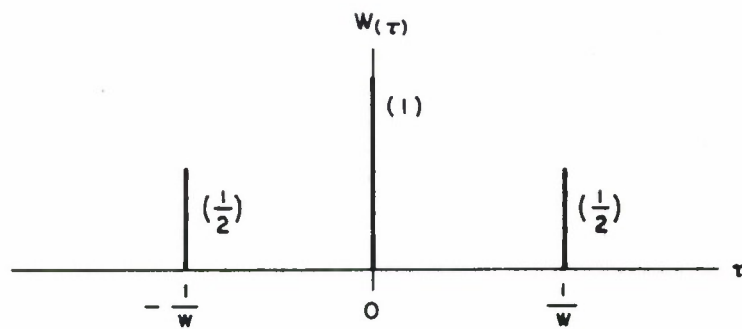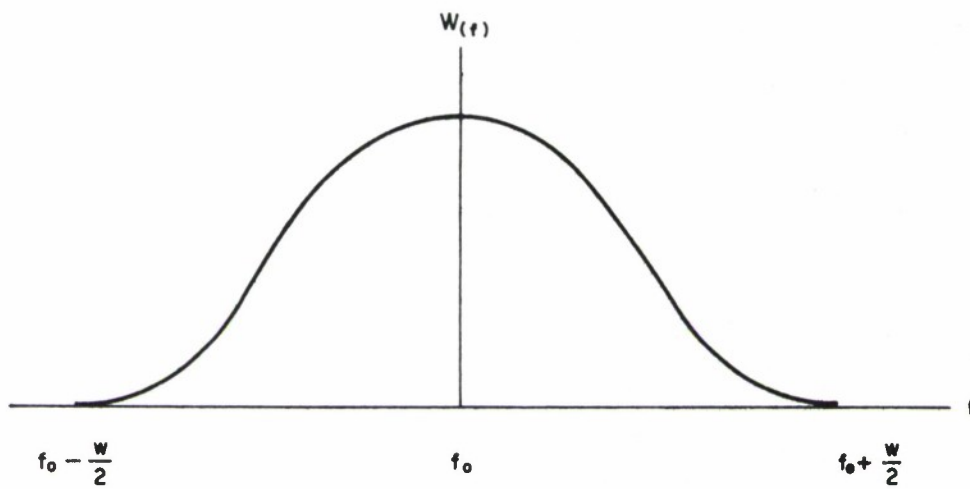
Figure 6.   Sidelobe Weighting Function 1 + Cosine

$$z_\ell = z_{1\ell} + z_{2\ell}$$

where

$$z_{1\ell} = \chi\left(\frac{\ell}{W}\right) = \frac{1}{2}\ \delta_{0,\ell-1} + \delta_{0,\ell} + \frac{1}{2}\delta_{0,\ell-1}$$

and

$$z_{2\ell} = +\frac{2j\Phi}{\pi^2}\ \frac{2\ell^2-4}{\ell^2(\ell-1)^2(\ell+1)^2} \qquad \text{for } \ell \geq 2$$

$$(48)$$

These calculations are best summarized in a table of the distortion samples $(Z_{2\ell})$ for both the weighted and unweighted case. Table I gives these results for the one plus cosine weighting and for $\Phi = .25$ radian.

Table I

Samples Of Distorted Output

$Z_{2\ell}$		
ℓ	unweighted	weighted
2	-22 db	-28 db
3	-40 db	-54 db
5	-48 db	-76 db

The distortion sidelobes are greatly attenuated by the sidelobe weighting because the distortion coefficients (λ_i) periodically change sign due to the $(-1)^\ell$ factor thus allowing the weighting function to

serve as a smoothing filter.* For large values of ℓ the samples drop off as $(1/\ell)^4$ rather than $(1/\ell)^2$ which occurs without weighting.

Although the distortion sidelobes are attenuated by weighting it is now shown that the quadratic distortion can broaden the main lobe to an extent much greater than the broadening due to the weighting alone. Thus there is a loss in resolution due to the quadratic phase distortion.

An exact analysis of the broadening effect would be tedious and possibly unrewarding. The following very approximate discussion is offered instead to give an intuitive understanding of the problem. The discussion relies on the analysis of linear FM signals by Key, et al.[8] Using the concept of stationary phase, all of the authors were able to analytically transform a signal with a large quadratic phase modulation. For signals with a large time bandwidth product the approximation is very good; for signals with a gaussian envelope the analysis is exact.

For this discussion, therefore, it is assumed that a 55 db Taylor function is approximately gaussian. The dependence of its transform on the amount of quadratic phase distortion is then determined.

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*Note: In this case the advantage of the various expansions is utilized because the weighted functions, χ, (Equation 22) are not orthogonal, whereas with weighting Ψ (Equation 21) are orthogonal.

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In Reference 8 it is shown that the envelope $|S(t)|$ a signal $s(t)$ with spectral magnitude

$$|S(f)| = \frac{e^{-\beta(f-f_o)^2}}{\sqrt{\pi}} \qquad (49)$$

and spectral phase

$$\angle S(f) = \theta(f) = c(f - f_o)^2 \qquad (50)$$

can be written as

$$|s(t)| = \frac{1}{(\beta^2 + c^2)} e^{\frac{-\pi\beta t^2}{\beta^2 + c^2}} \qquad (51)$$

The constant β determines the degree of weighting and the constant c determines amount of quadratic phase. Let

$$\beta = \frac{B}{W^2}$$

and

$$c = \frac{4\Phi}{W^2}$$

then the resolution of the gaussian signal, $s(t)$, defined as half the pulse width at the -55 db level, is

$$Res = \frac{1}{W} \left[\frac{55}{20\pi \log e} \right]^{\frac{1}{2}} \left[\frac{B^2 + [4\Phi]^2}{B} \right]^{\frac{1}{2}} \qquad (52)$$

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For small Φ there is almost no loss in resolution, while for large Φ the resolution is approximately

$$\text{Resolution} \approx \sqrt{\frac{55 \cdot 16}{20 \ \pi \ B \ \log \ e}} \quad \frac{\Phi}{W} \tag{53}$$

If, for example, B = 8 the weighting function is down 20 db at the edge of the frequency band W and the -55 db pulse width is $\frac{2\Phi}{W}$.

NUMERICAL CALCULATIONS

The results of numerically computing the output time signal, $z(t)$ for the case where a wideband flat spectrum input time signal is distorted by a quadratic phase function and Taylor weighting are presented in this section. As noted earlier, a large time bandwidth linear FM pulse or an ideal bandpass impulse would have a spectrum such that

$$\left| X(f) \right| = 1 \qquad \text{for } f_o - \frac{W}{2} \leq f \leq f_o + \frac{W}{2} \tag{54}$$

The integral in Equation 4 was evaluated using traditional numerical analysis techniques for the quadratic distortion given in Equation 26.

The following table indicates the figure number, the degree of weighting, and the amount of quadratic phase.

Table II

Figure Numbers

db weighting \ Φ	0.0	0.2	0.4	0.8	1.6	3.0	6.0
55	7	8	9	10	11	12	13
35	14	-	-	15	-	-	16

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The output time signal in Figures 7 - 16 are plotted with the peak of the main lobe normalized to one. Besides the loss in signal-to-noise ratio due to Taylor weighting there is an additional loss in signal-to-noise ratio due to the phase distortion. The quadratic phase distortion effectively attenuates the signal level because the signal energy is dispersed. This loss is plotted in Figure 17 A, B for no weighting and 55 db and 35 db Taylor weighting.

From the plots of the output time signal, it can be seen that there is a loss in signal resolution. This effect is measured in two ways. It is asserted that the most meaningful measure of resolution is the width of the main lobe at a level equal to the largest sidelobe. The distance from the center of the main lobe to that point is plotted in Figure 18 for the cases where the signal is weighted by a 55 db or 35 db Taylor function and where it is unweighted. The more conventional, but less meaningful, 3 db resolution is plotted in Figure 19 for the case of 55 db Taylor weighting and no weighting. The curve for the loss in resolution measured at the 3 db level is similar to the curve predicted by Equation 52. The curve in Figure 18 has different characteristics because of the complex interactions of the near-in sidelobes.

34
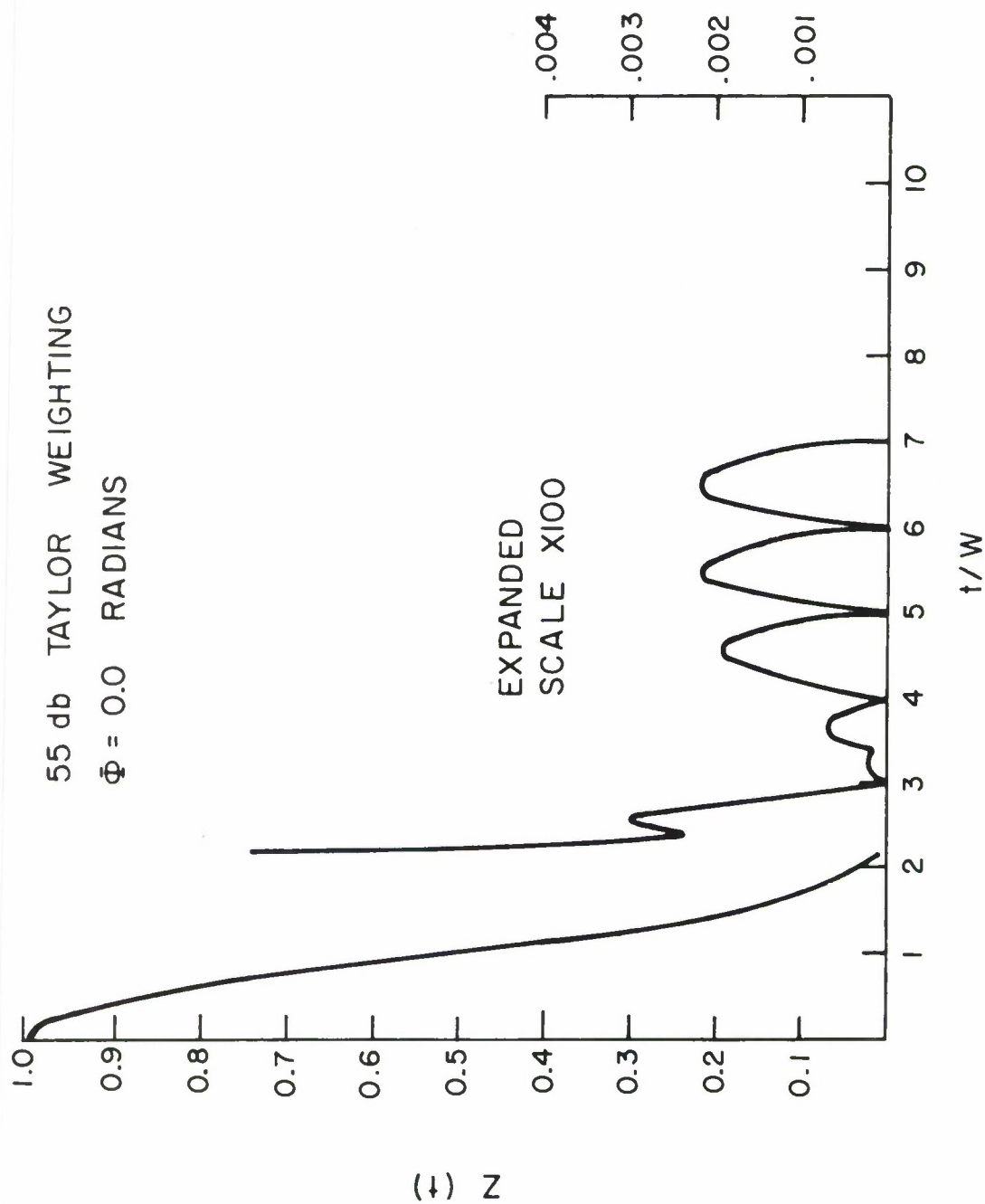
Figure 7. Output Time Signal Quadratic Phase Distortion
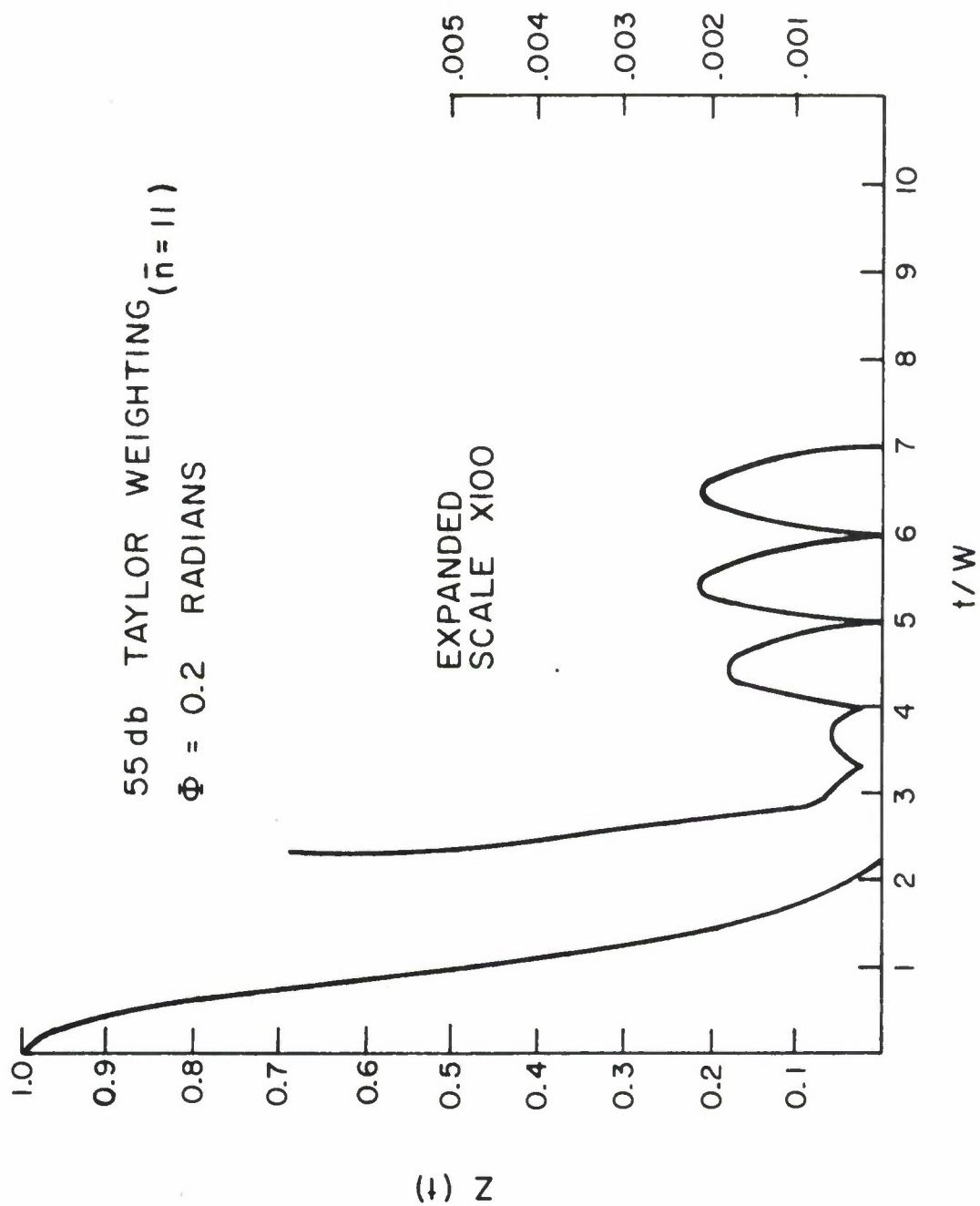
Figure 8. Output Time Signal Quadratic Phase Distortion

IA-22,317
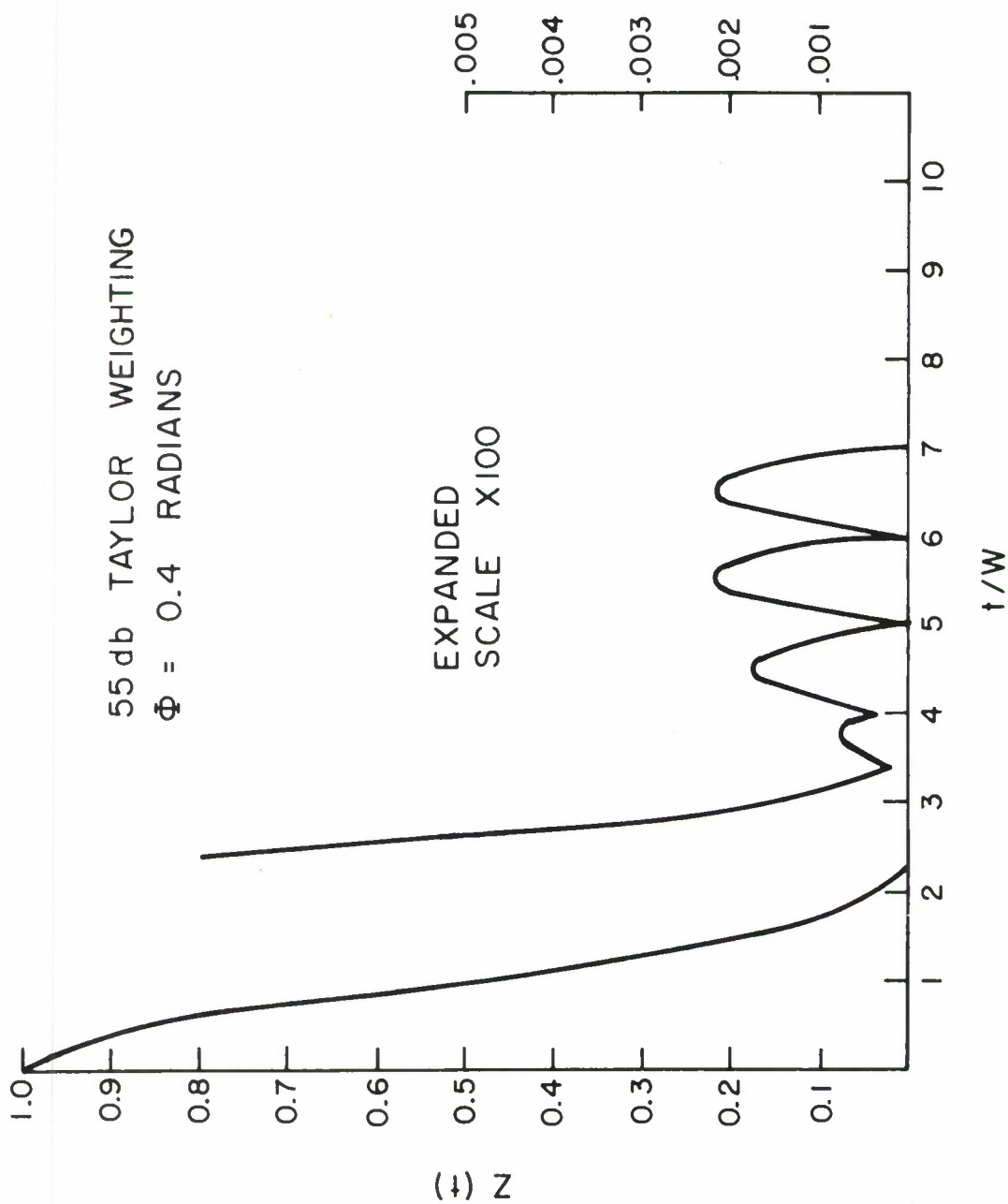
Figure 9. Output Time Signal Quadratic Phase Distortion

55 db TAYLOR WEIGHTING
Φ = 0.4 RADIANS

EXPANDED SCALE X100

t/W

Z (t)
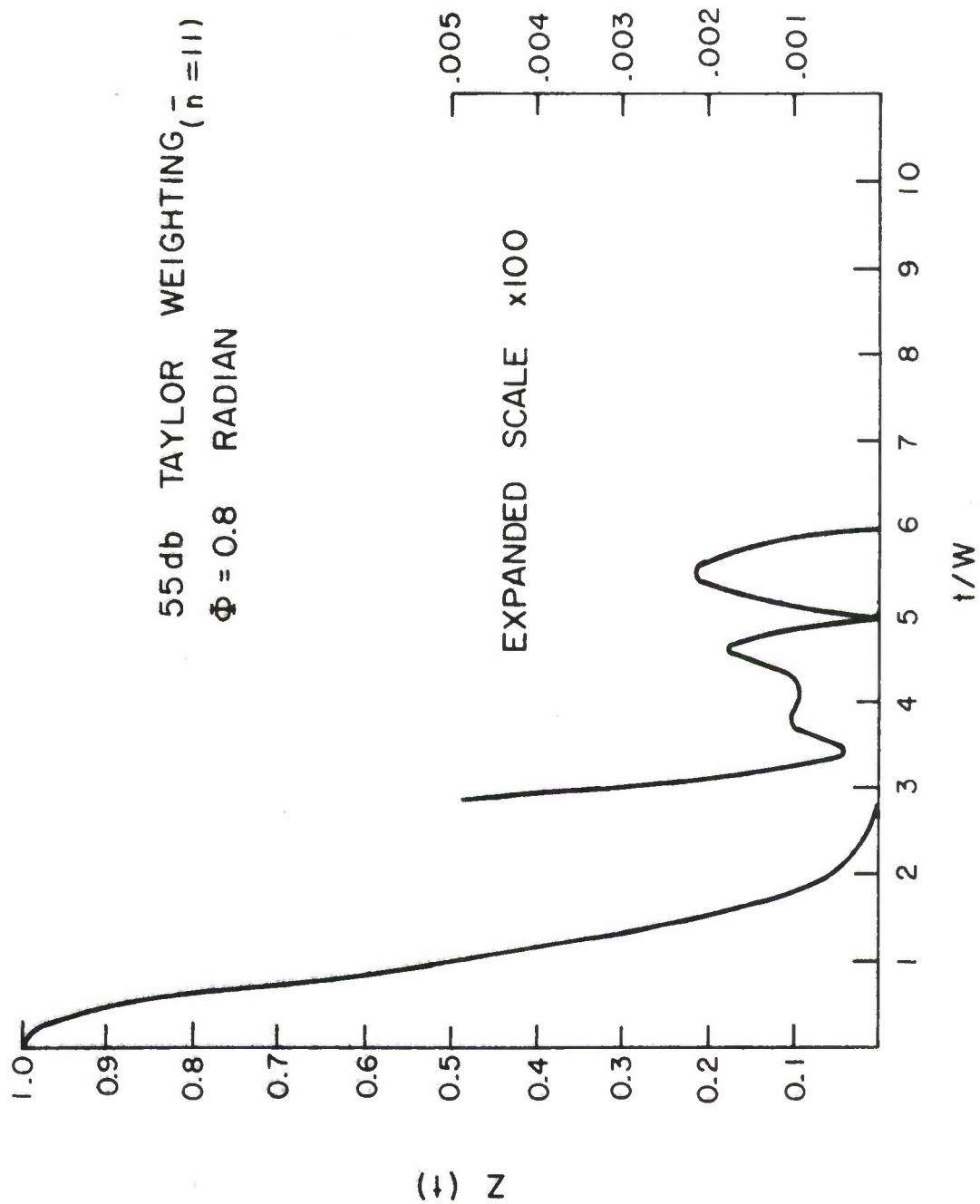
Figure 10. Output Time Signal Quadratic Phase Distortion
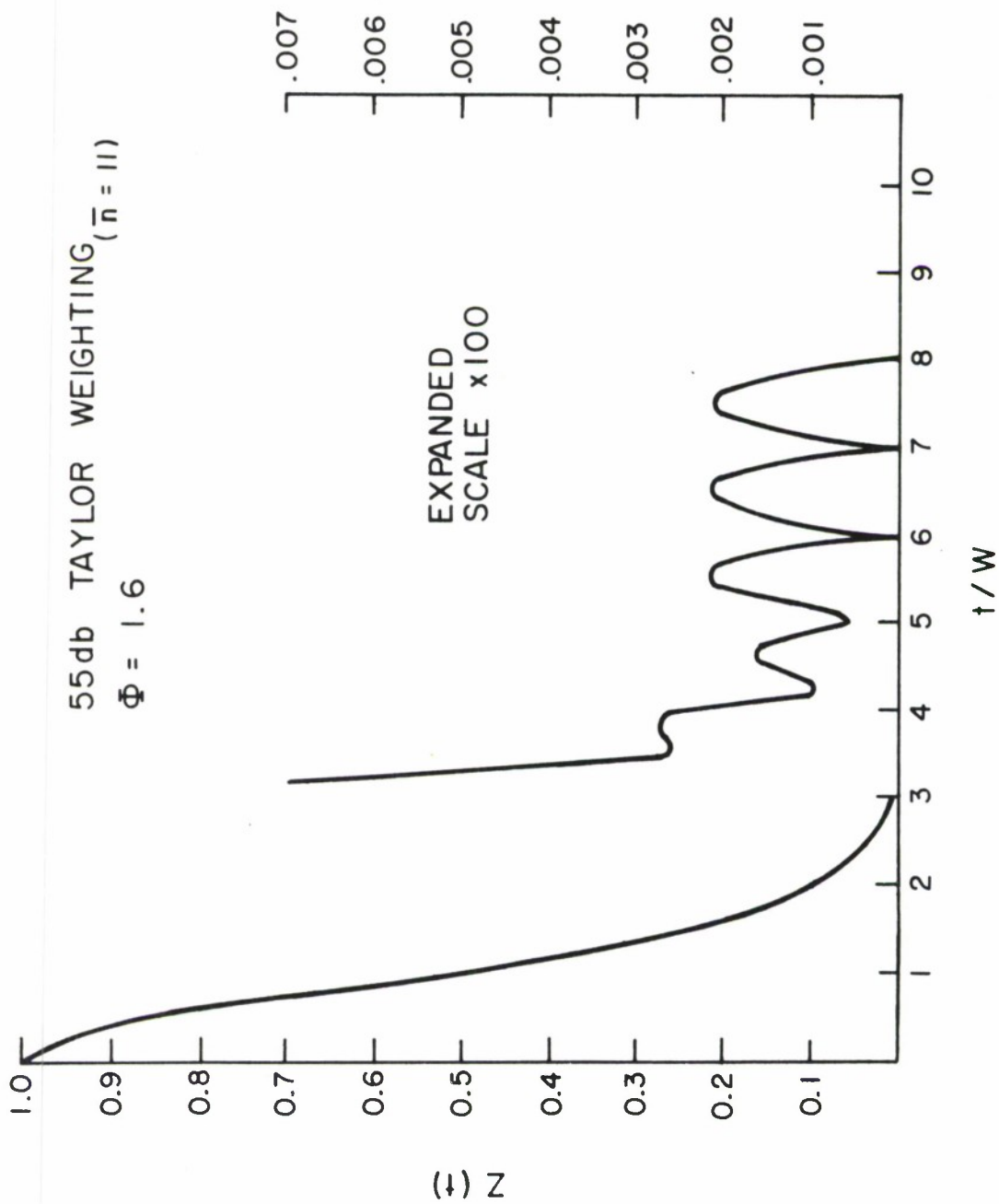
Figure 11. Output Time Signal Quadratic Phase Distortion

39
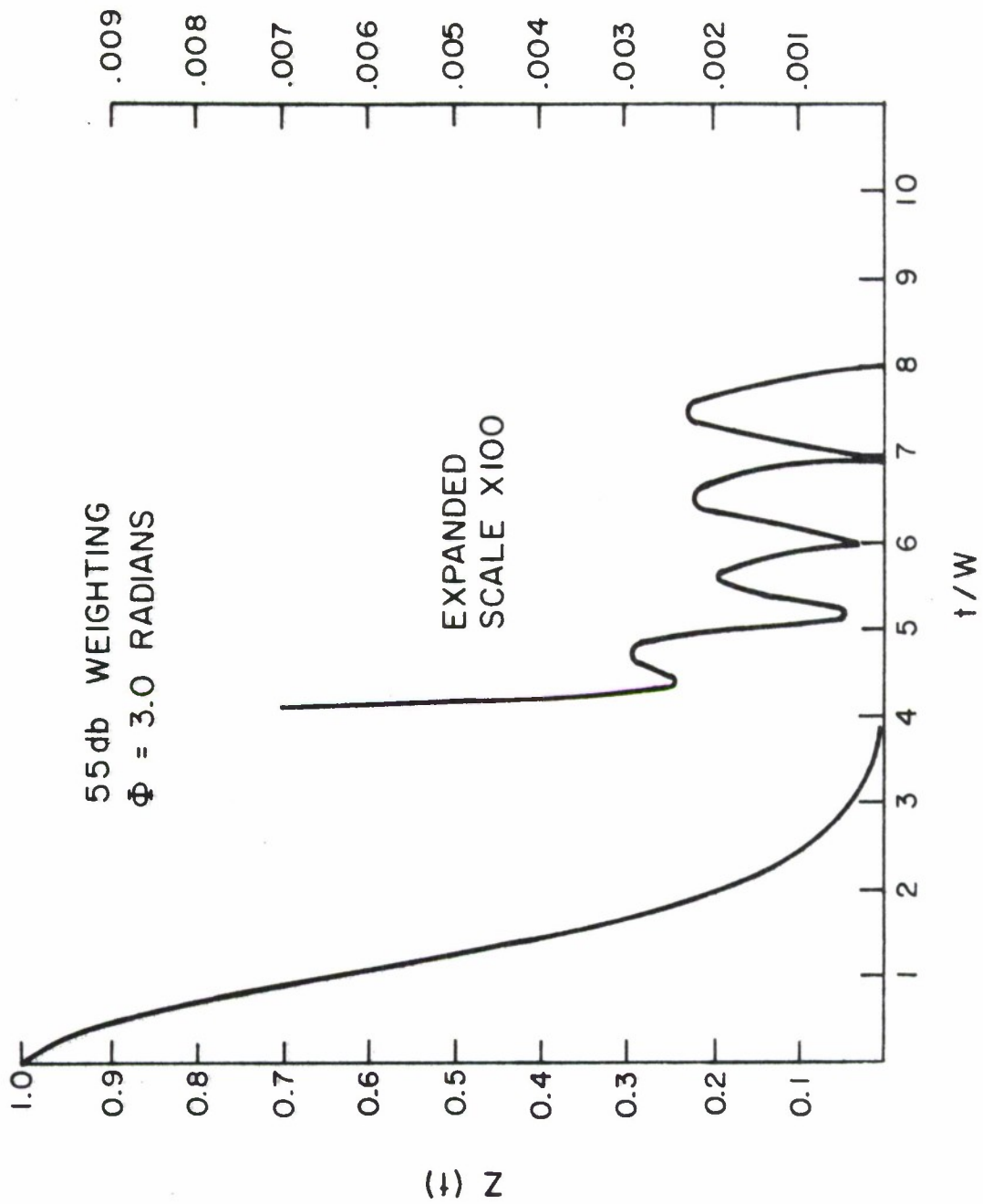
Figure 12. Output Time Signal Quadratic Phase Distortion

55 db WEIGHTING
Φ = 3.0 RADIANS

EXPANDED
SCALE X100

t/W

Z(t)

IA-22,319
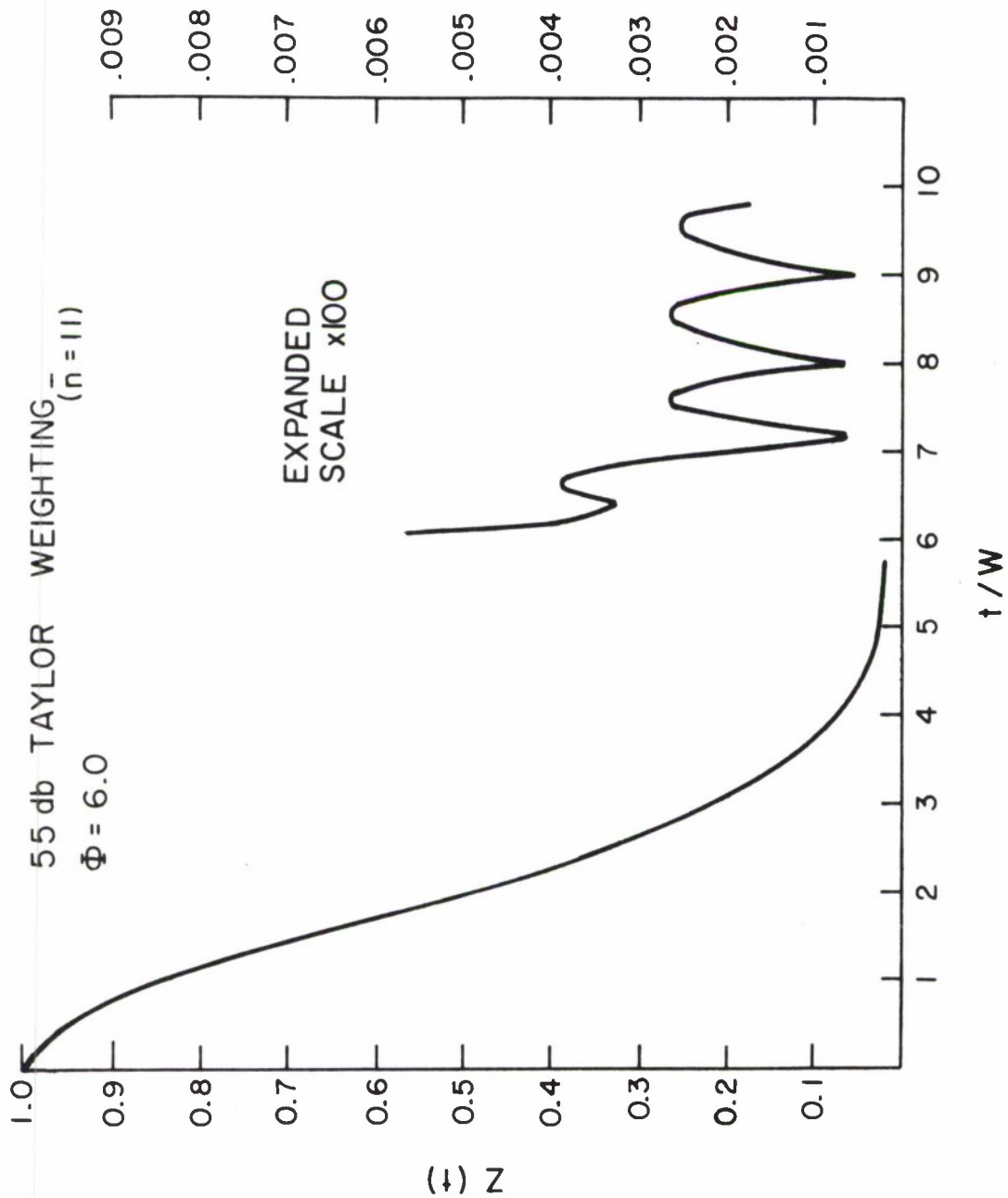
Figure 13. Output Time Signal Quadratic Phase Distortion
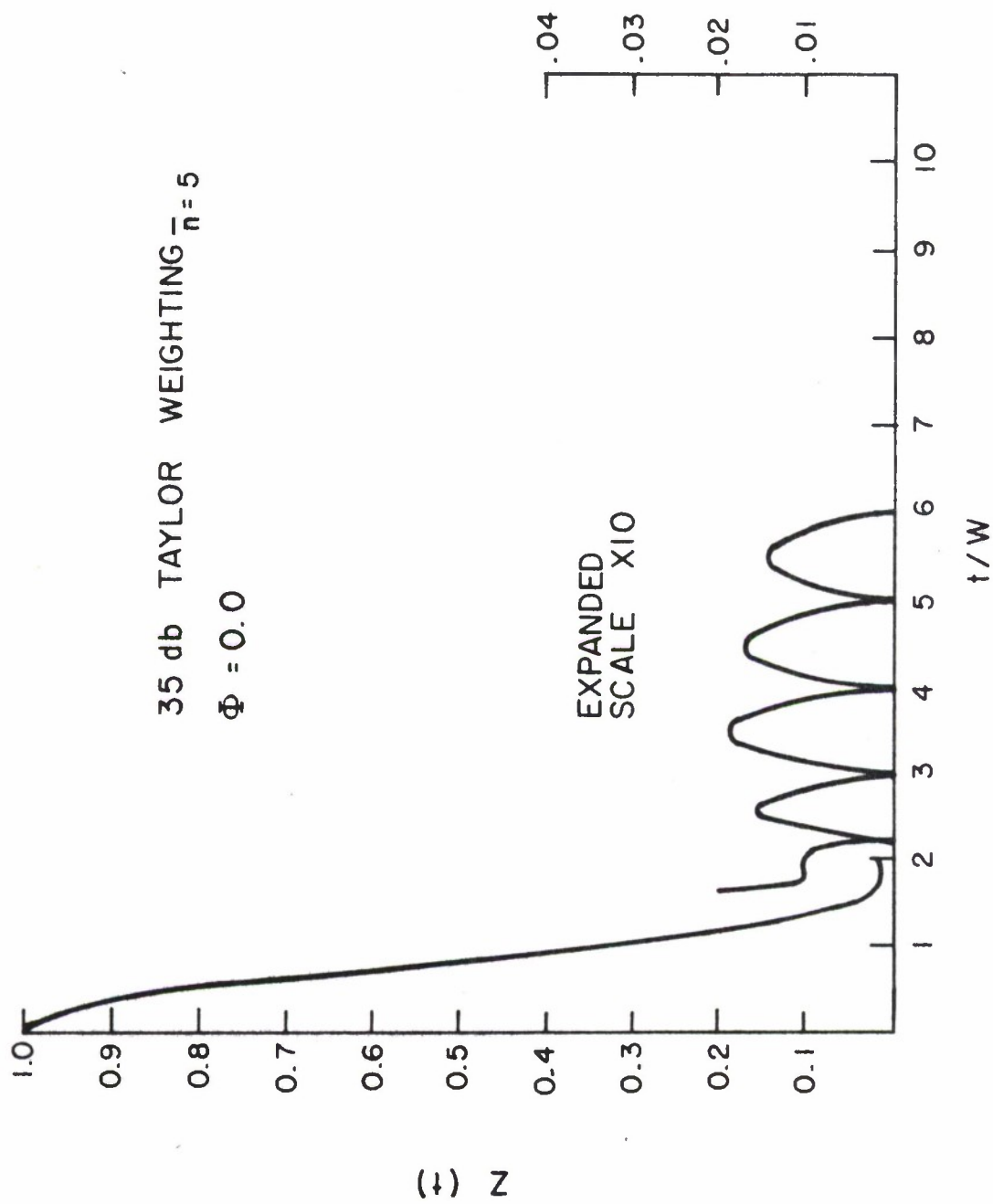
Figure 14. Output Time Signal No Phase Distortion

35 db TAYLOR WEIGHTING $\bar{n}=5$

$\Phi = 0.0$

EXPANDED SCALE X10

t/W

Z (t)
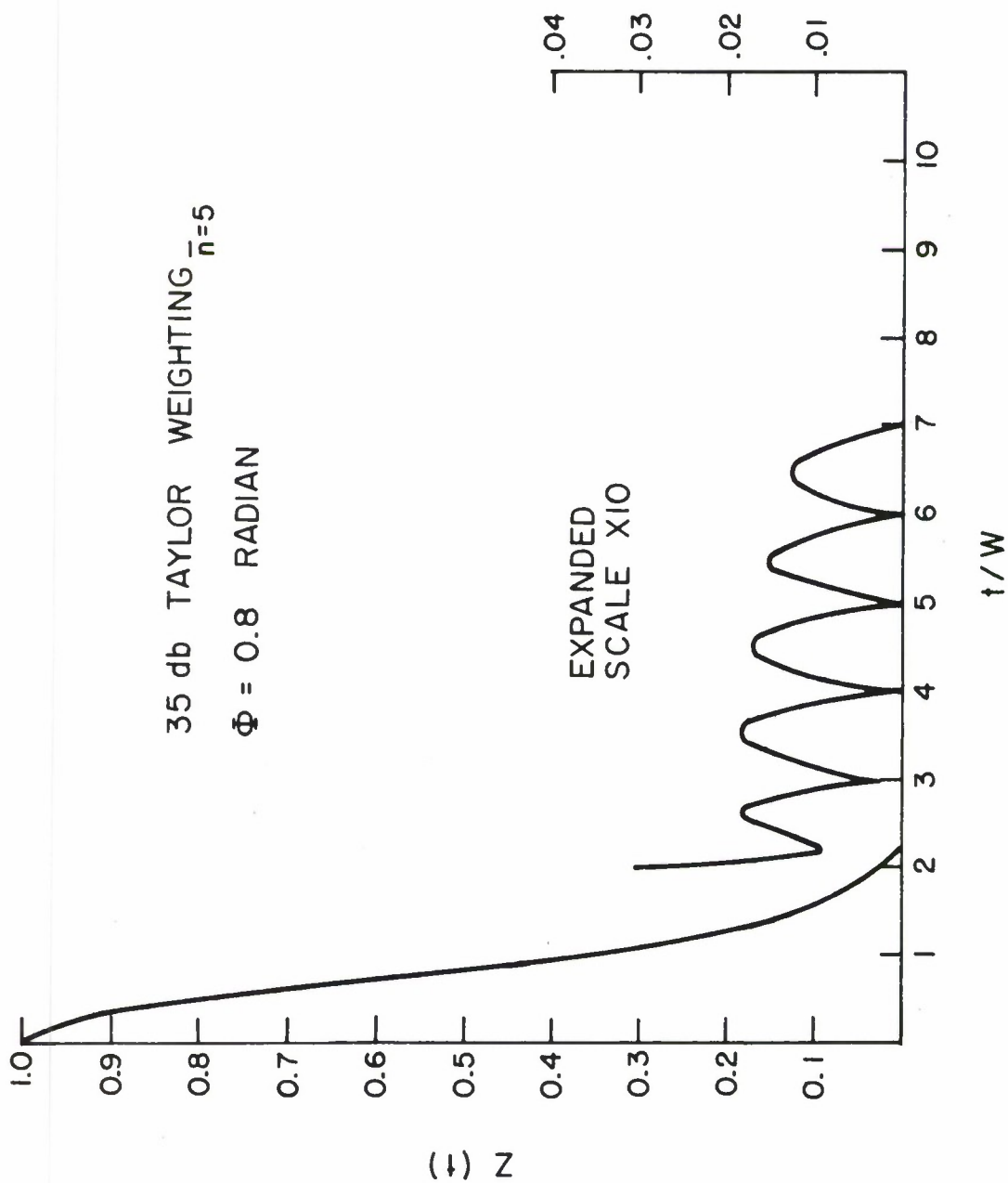
Figure 15. Output Time Signal Quadratic Phase Distortion

35 db TAYLOR WEIGHTING $\bar{n}=5$

$\Phi = 0.8$ RADIAN

EXPANDED SCALE X10

t/W

$Z(t)$
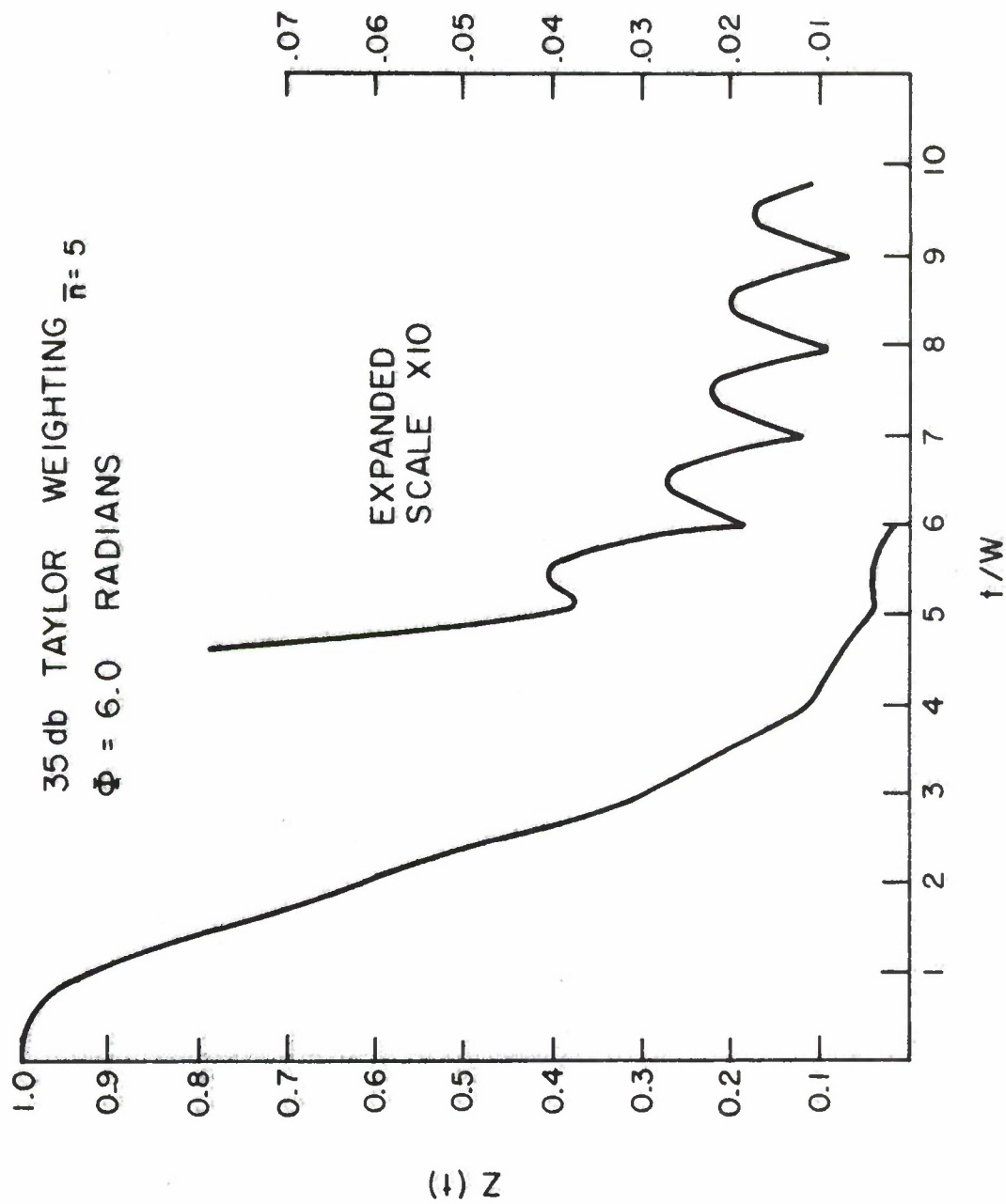
Figure 16. Output Time Signal Quadratic Phase Distortion

35 db TAYLOR WEIGHTING $\bar{n} = 5$
$\Phi = 6.0$ RADIANS

EXPANDED SCALE X10

t/W

Z(t)

VOLTS

SIGNAL ATTENUATION

(SMALL SCALE)

△ NO WEIGHTING
● 35 db TAYLOR WGT
X 55 db TAYLOR WGT

10

.9

.8

.7   3 db

.6

.5   6 db

.4

.3

.2   14 db

.1

55 db WGT.

35 db WGT.

NO WGT.

1   2   3   4   5   6

Φ RADIANS

IA−22,330
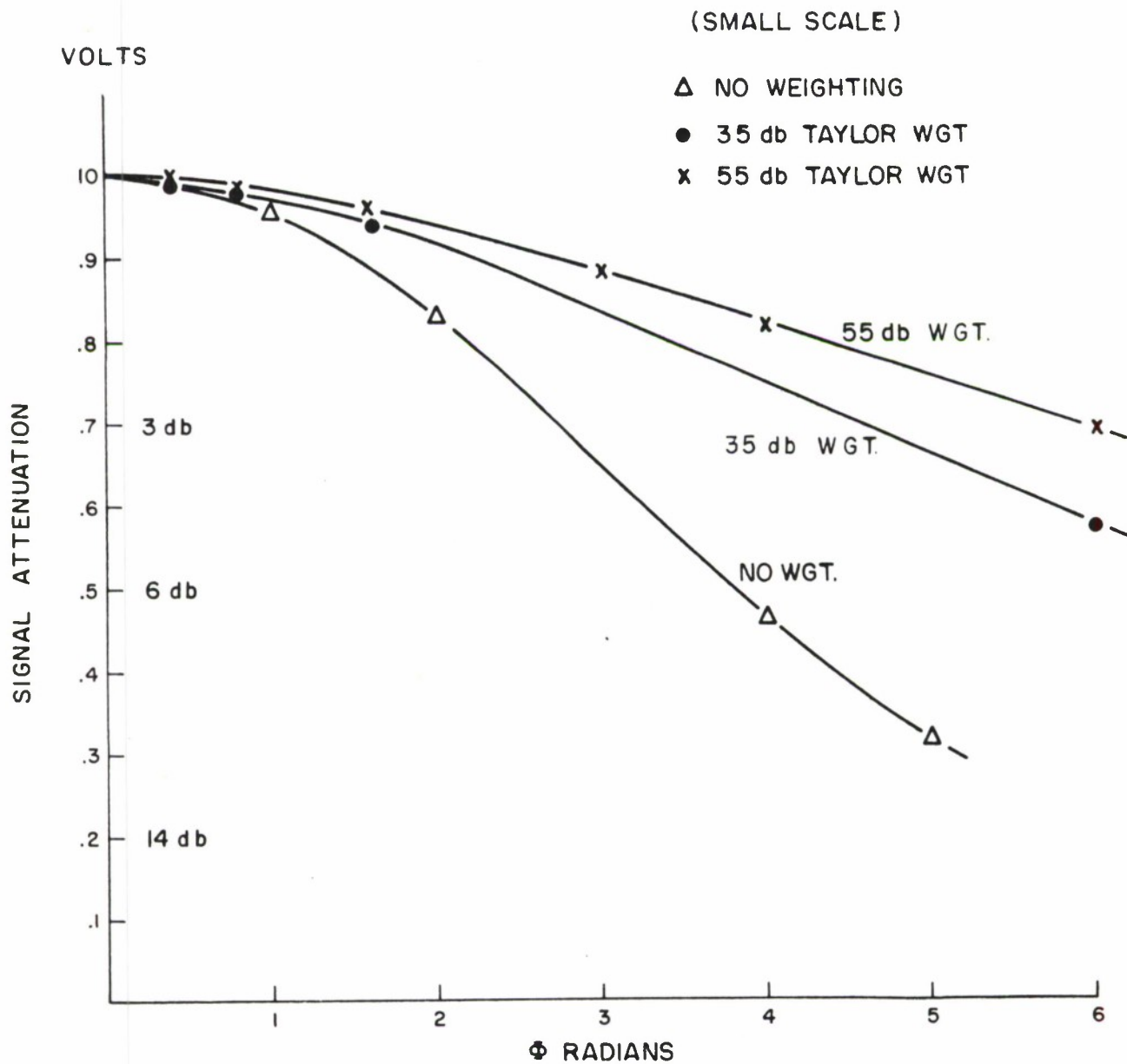
Figure 17a. Attenuation vs Quadratic Phase Distortion
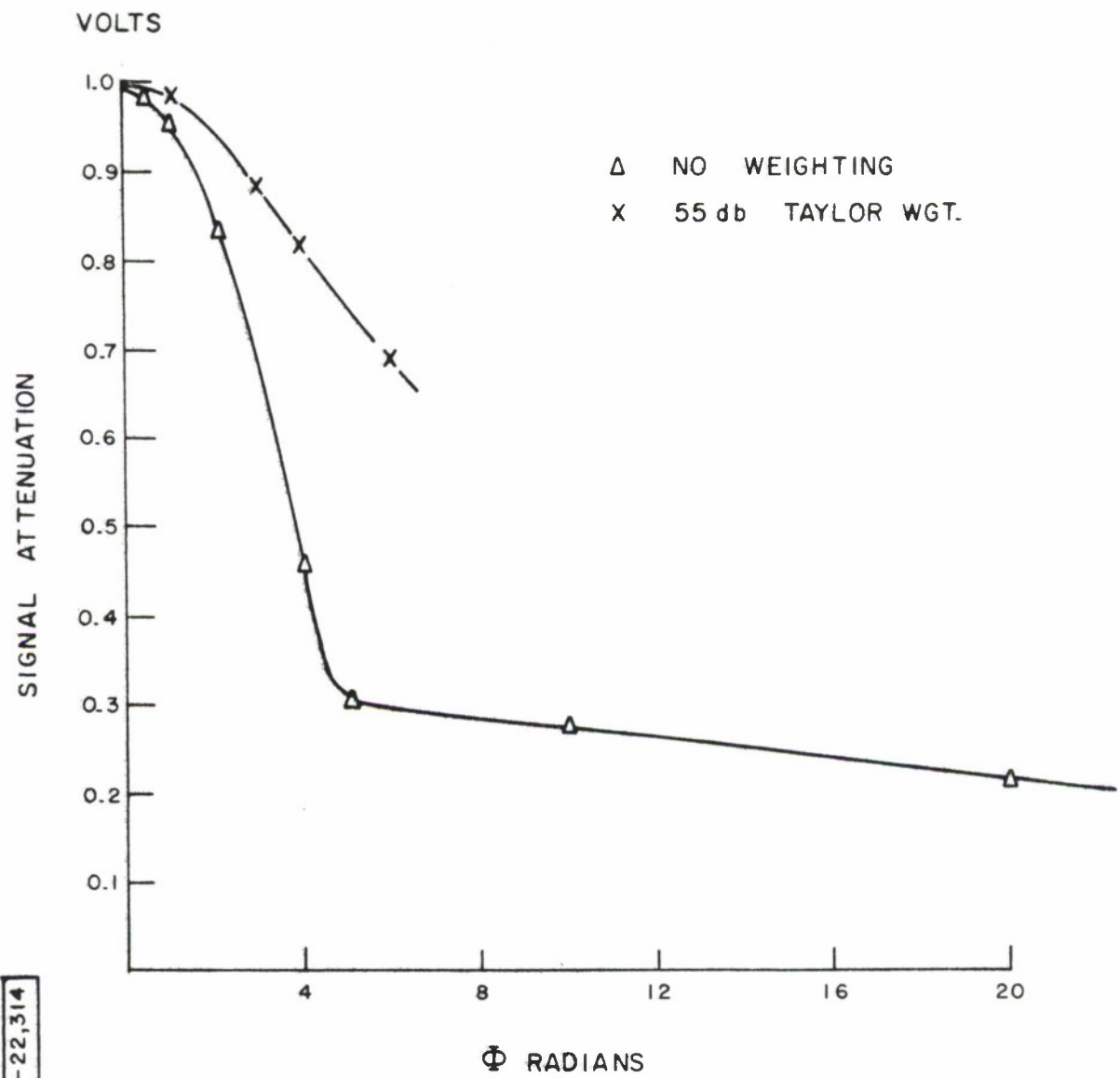
Figure 17b. Attenuation vs Quadratic Phase Distortion

IA—22,314
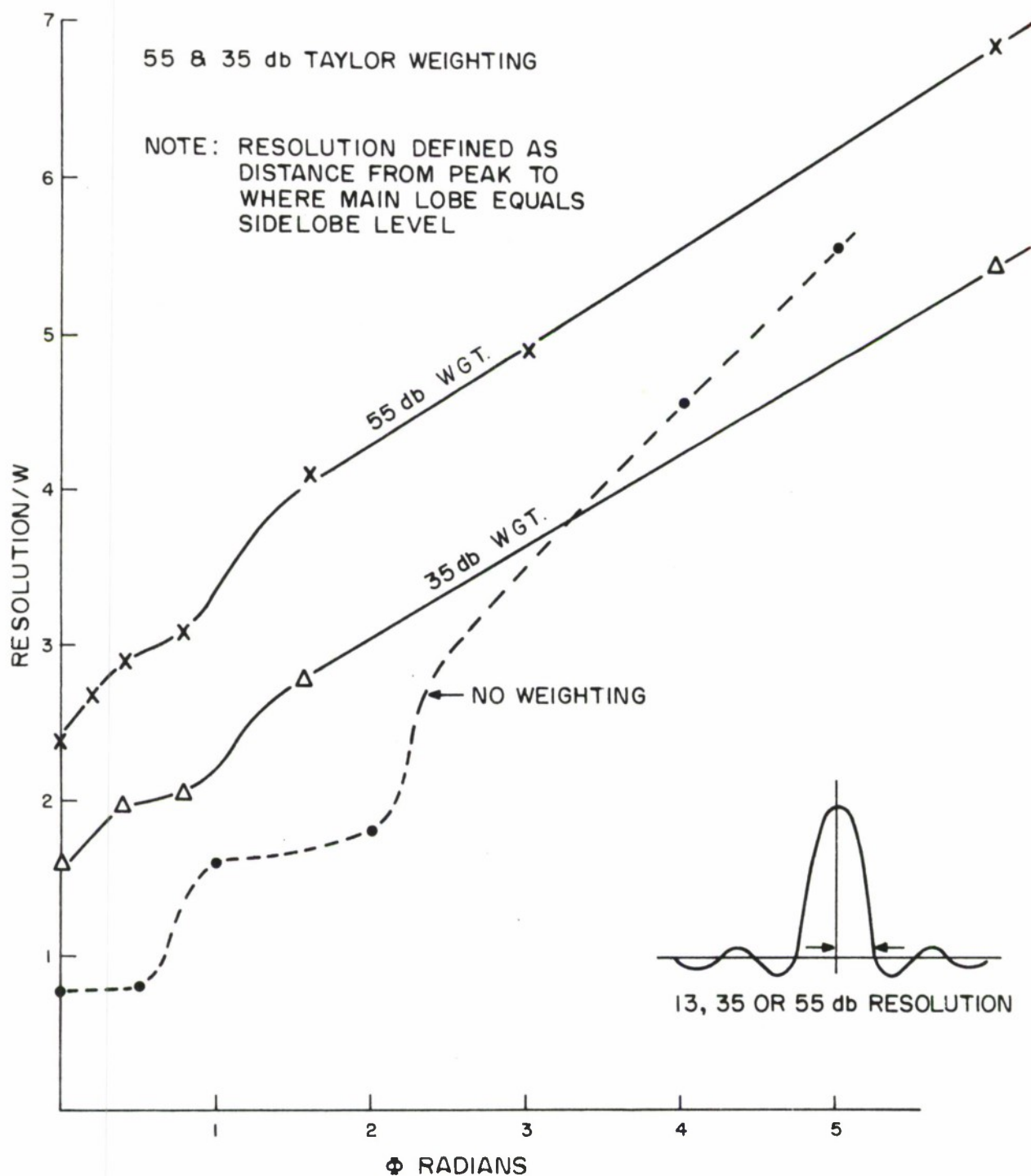
Figure 18. Resolution vs Quadratic Phase Distortion
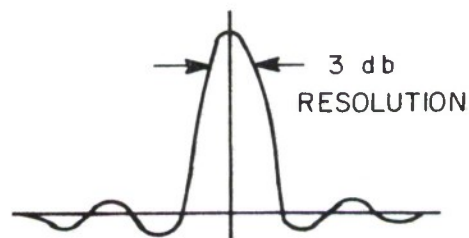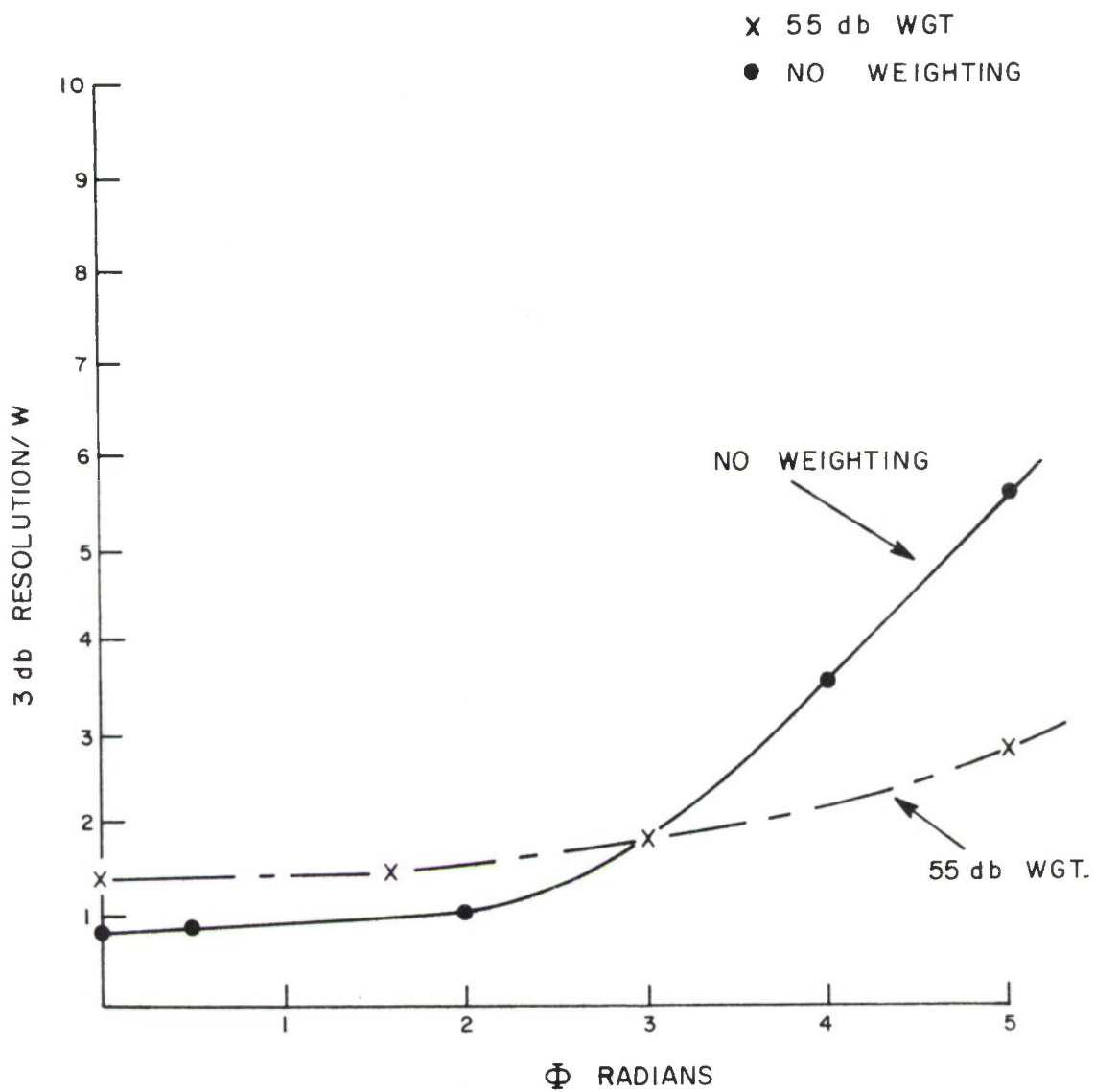
Figure 19. 3 db Resolution vs Quadratic Phase Distortion

SECTION V

APPLICATION TO IONOSPHERIC DISPERSION

The group delay of a signal propagating through the ionosphere
is a function of frequency, i.e., its phase characteristics are non-
linear. Using a simple electromagnetic model for the propagation, a
wave equation can be found. It indicates that the phase -vs- frequency
characteristic has a large quadratic term, but that higher-order power
series terms are unimportant. Thus, the study of ionospheric distor-
tion is reduced to determining the functional dependence of Φ on the
physical parameters of the ionosphere. The analysis only considers
propagation above plasma frequency (VHF to X-band). For this band the
quadratic phase is proportional to the integrated electron density.
There are many papers and texts dealing with ionospheric propagation;
see, for example, Reference 9. Because there are so many, only
the results will be given.

An extension of a propagation analysis to include gross inhomo-
geneities, dense electron environment and time-varying effects are
well beyond the scope of this paper. The application of this study to
propagation below the plasma frequency (HF), while a very interesting
area, has not been included.

The ionosphere consists of free electrons; therefore, it affects
the propagation of electromagnetic energy. By considering the number
of these electrons, their charge and mass and by making simplifying

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assumptions about their interactions, a wave equation can be derived. The solutions of the wave equation are in such a form that it is possible to relate the input and output signals of the media by a linear relationship. The linear relationship is a convolution of the input time signal and the impulse response of the media.

It is assumed that the free electrons in the ionosphere do not interact and they they are statically and homogeneously distributed throughout a local region described by the wave equation. Using Maxwell's equations and the force on an electron, the following wave equation can be derived:

$$\frac{d^2 E(z)}{dz^2} + \frac{\omega^2}{c^2} \left[1 - \frac{N e^2}{\epsilon_o m \omega^2} \right] E(z) = 0 \tag{55}$$

where

$E(z)$ is the electromagnetic field

z is distance

$\omega = 2\pi f$ is frequency in radians/sec

e is electron charge

m is electron mass

N is electron density

c is speed of light .

Solving the wave equation and identifying the equivalent filter characteristic it is found that

$$\left| I(f) \right| = 1 \tag{56}$$

$$\angle I(f) = \phi(f) = \int_0^Z \cdot \frac{2\pi f}{c} \sqrt{1 - \frac{81 N(z)}{f^2}} \ dz$$

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where   Z is the total path length

               N is assumed to vary slowly with Z

For frequencies well above the plasma frequency this equation
may be expanded in a binomial series and only terms below second order
retained. Thus, if

$$f \gg 9 \sqrt{N}$$

$$\phi(f) \approx \frac{2\pi f Z}{c} - \frac{81\pi}{cf} \int_0^Z N(z)dz \qquad (57)$$

The first term represents the natural delay $\tau = Z/c$. The second term
represents a distortion. Expansion of this term in a Taylor series
reveals that the cubic and higher order terms are negligible. The
first order term is a delay which causes range errors but does not
distort the pulse. The second order term can be written in the general
quadratic form given in Equation 26, where

$$\Phi = \frac{81\pi \left[\frac{W}{f_o}\right]^2}{8\, c\, f_o} \int_0^Z N(z)dz \qquad (58)$$

For a 10 percent bandwidth signal $\frac{W}{f_o} = .1$ the expression is

$$\Phi \approx \frac{10^{-9}}{f_o} \int_0^Z N(z)dz \qquad (59)$$

Table III gives Φ for an integrated electron density which
is (1) a typical worst case density, and (2) a typical median density.

This ionospheric data was extracted from Reference 10, but because the electron density is a strong function of many factors, such as sun spot number, elevation angle, and time of day, the reader must be careful in interpreting exactly what is meant by a worst case or median ionosphere.

Table III

MAX Quadratic Phase Shift (Φ In Radians)
For 10 Percent Bandwidth

| Band | f_o | Φ for Two-Way Integrated Electron Density of | |
		4×10^{18} e/m^2	2×10^{17} e/m^2
L	1 Gc	4.0 (Figure 12)	0.2 (Figure 8)
C.	5 Gc	0.08 (Figure 10)	0.04
X	10 Gc	0.04 (Figure 9)	0.02

Note: The figure numbers refer to the results of the numerical calculations made for the closest value of quadratic phase using 55 db weighting.

SECTION VI

APPLICATION TO OTHER PROBLEMS

There are several problem areas which either have quadratic
phase distortion or closely related distortion which can be analyzed
directly by a simple modification of the previous work. The problems
discussed in this section are:

1.   Time Dilation

2.   Waveguide Dispersion-Quadratic Effects

3.   Frequency-Dependent Radar Cross-Section

4.   Waveguide — Cubic Phase Function Analysis.

TIME DILATION

A constant velocity target distorts a signal in a manner ref-
erred to as "time dilation". The distortion is a rescaling of the time
axis, and therefore also the frequency axis. The first order effect
is a shifting of the center frequency of the signal — the common
"doppler effect". One of the second-order effects is a change in the
signal bandwidth; however, this distortion results in a loss in signal-
to-noise ratio which is usually insignificant. For phase modulated
signals the time dilation distorts the signal in a manner similar to
a dispersive filter. A linear FM signal — a type of phase-modulated
signal — is distorted by the introduction of additional quadratic phase
(i.e., slope of linear FM changes). For large time bandwidth signals
the distortion can result in very significant loss of signal-to-noise

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ratio and resolution.

The amount of quadratic phase as a function of target velocity and signal parameters is calculated to determine the degree of the distortion and the accuracy that is required in measuring the target velocity to apply corrective techniques.

An accelerating target introduces a quadratic phase distortion on any signal. Therefore, its effects are also discussed in this section.

Assume a received FM signal of duration T :

$$s(t) = \sin \left[\frac{\pi W}{T} (t - \alpha(t))^2 + w_c (t - \alpha(t)) \right] \qquad (60)$$

The delay $\alpha(t)$ is a time-dependent delay increment due to a single moving target:

$$\alpha(t) = \frac{2R_o}{c} + \frac{2\dot{R}t}{c} + \frac{\ddot{R}t^2}{c} \qquad (61)$$

The receiver mixes the received signal with a local oscillator (LO) with the same characteristics as the transmitted signal.

$$LO(t) = \sin \left[\frac{\pi W}{T} t^2 + w_c t \right] \qquad \text{for } -\frac{T}{2} < t < \frac{T}{2} \qquad (62)$$

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The low pass output of the mixing operation is

$$r(t) = \sin(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4) \tag{63}$$

The a_0 term is simply a phase shift which is ignored in the following discussion. The a_1 term is

$$a_1 = \frac{2\dot{R}\,\omega_c}{c} + \frac{4\pi W\,R_0}{cT} - \frac{8\pi W\,R_0\,\dot{R}}{Tc^2} \tag{64}$$

Thus, the a_1 term contains both target range and range-rate information (usual range-doppler coupling). The last term (Equation 64) is ignored because it represents an error which is usually very small.

The significant distortion is caused by the quadratic term a_2. This term has value

$$a_2 = -\frac{\omega_c \ddot{R}}{c} + \frac{4\pi W\dot{R}}{cT} + \frac{4\pi W R_0 \ddot{R}}{c^2 T} \tag{65}$$

For most linear FM signals the $\dot{R}$ term dominates. The maximum quadratic phase ϕ is given by

$$\phi = a_2 \left[\frac{T}{2}\right]^2 = \frac{\pi W T \dot{R}}{c} \tag{66}$$

For a time-bandwidth product of a million (say $T = 1$ millisecond, $W = 1$ GHz, $c = 3 \times 10^8$ m/sec).

$$\phi = 10^{-2}\,\dot{R} \tag{67}$$

55

Target velocities as large as 1,000 meters/sec are not uncommon, therefore it is possible for $\ddot{\phi}$ to be as large as 10 radians (see Figure 5 for an unweighted distorted signal with $\ddot{\phi} = 20$ — there are no plots of weighted signals with so much distortion). Even a target velocity which is an order of magnitude less (100 meters/sec) would produce 1.0 radians of quadratic phase (see Figure 11 for weighted distorted signal). If the quadratic phase is to be below .2 radian (considering loss in resolution, see Figure 8) the range rate $\dot{R}$ must be measured with an accuracy of better than 20 meters/sec.

For most applications the target acceleration does not introduce significant distortion. For example, for a millisecond X-band signal, the acceleration must be known within 2,000 meters/sec^2. However, if a signal is to be coherently integrated for about one second, the acceleration must be known within .002 meters/sec^2 — a stringent requirement.

WAVEGUIDE DISPERSION - QUADRATIC EFFECTS

The phase characteristic of waveguide has almost the same functional form as the dispersive ionosphere characteristic. If the ionosphere electron density is a constant, the relationships are identical. Waveguide, however, has a much higher cut-off frequency than the ionosphere. For example, typical X-band waveguide may cut off at 6 GHz while the ionosphere may cut off as low as 10 MHz. The quadratic phase function for waveguide is evaluated in this section; an analysis which includes cubic phase effects is given later in this Section.

The phase characteristic of waveguide can be written as

$$\phi(f) = \frac{2\pi f Z}{c} \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \qquad (68)$$

The magnitude of the quadratic term of a Taylor series expansion is found by evaluating the second derivative of the phase function at the signal center frequency f_o. The maximum quadratic phase Φ is found to be

$$\Phi = \frac{\pi W^2 Z X^2}{4 f_o c (1 - X^2)^{3/2}} \qquad (69)$$

where $X = f_c / f_o$ = relative cut-off frequency.

The "dispersion" (total change in group delay across the band) is given by

$$\Delta = \frac{4\Phi}{\pi W} = \left[\frac{W}{f_o}\right] \cdot \frac{Z X^2}{c (1 - X^2)^{3/2}} \qquad (70)$$

Thus, the dispersion is a linear function of the percent bandwidth and the waveguide length, but is a complicated function of the relative cut-off frequency. Because the dispersion is independent of center frequency, f_o, the distortion effects get worse as f_o increases if the percent bandwidth is held fixed $(\frac{W}{f_o})$. For example, a nanosecond of dispersion is insignificant in an L-band system with 100 Mc bandwidth but is very important in an X-band system

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with 1000 Mc bandwidth. To relate the distortion to the previous results, a table is provided giving values of ϕ for various lengths and relative cut-off frequencies. The information in the table arbitrarily assumes a center frequency of 10 GHz and a bandwidth of 1 GHz. See Table II in Section IV for the figure numbers of the graphs showing the distorted signals.

Table IV

Quadratic Waveguide Dispersion

ϕ in Radians for f_o = 10 GHz, W = 1 GHz

Z	$\dfrac{f_c}{f_o}$ .1	.5	.6	.7
1 meter	.025	.1	.15	.35
10 meters	.25	1.0	1.5	3.5
50 meters	1.25	5.0	7.5	17.5

From this data it can be seen that for even a modest length of waveguide serious distortion can occur for standard waveguide (e.g., X-band waveguide WR 112 has f_c = 5.26 GHz).

FREQUENCY-DEPENDENT RADAR CROSS-SECTION

An example of non-harmonic amplitude distortion is frequency-dependent radar cross-section of various types of scatterers. Using the geometric theory of diffraction, the radar return from a plate scatterer can be calculated as if the signal had passed through a filter with the following frequency characteristic

$$\left|S(f)\right| = 2\pi f$$

$$\angle \ S(f) = \phi(f) = 0 \tag{71}$$

Other types of scatterers such as cylinders, edges and points also have similar, but different, frequency characteristics.

The analysis procedure discussed in Section III for a quadratic phase function is applicable to such a problem. More important, however, the sidelobe weighting analysis of Section IV shows that the major effect of this distortion is the loss of resolution because the main lobe is widened.

Following the analysis pattern almost directly in Equations 32 through 38, the Fourier coefficients of the signal λ_i and therefore the paired echo coefficients of the output time signal are found to be

$$y(\frac{i}{W}) = \lambda_i = \int\limits_{f_o-\frac{W}{2}}^{f_o+\frac{W}{2}} S(f) \ e^{\frac{j2\pi fi}{W}} \ df \tag{72}$$

$$y(\frac{i}{W}) = \lambda_i = \left[\frac{jd\psi(t)}{dt} + \psi(t)\right]_{t \ = \ \frac{i}{W}}$$

$$= j \ (\frac{W}{f_o}) \ \frac{(-1)^{i+1}}{i}$$

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Three conclusions can be drawn from the expression for the paired echo coefficients:

1. The effect is proportional to the percent bandwidth.

2. The Fourier coefficients drop off inversely as their number.

3. The coefficients alternate in sign so that weighting will reduce their effect, but not to the same degree as for a quadratic phase function.

In practice, the percent bandwidth is usually less than one, therefore this type of distortion is not as important as the quadratic phase distortion which could become severe (see waveguide dispersion).

WAVEGUIDE - CUBIC PHASE ANALYSIS

Dispersive media, such as waveguide when operated near the cut-off frequency, introduces phase distortions which may include non-negligible terms of higher than second order. VHF propagation through the ionosphere could also have this characteristic. In this section the coefficient of the cubic phase error term for waveguide is calculated and its effect compared to the effect of the quadratic term. Because it is possible to easily correct for quadratic distortion in a linear FM system (by a flexible slope) the cubic term may be a limiting factor in determining the system resolution. Therefore, the loss in resolution is a function of the cubic term in certain applications.

The dispersive waveguide characteristic is given by

$$\phi(f) = \frac{2\pi f Z}{c} \sqrt{1 - (f_c/f)^2}$$

The cubic term of a Taylor series expansion is found by taking the third derivative of $\phi(f)$ and evaluating it at the center frequency f_o. Thus

$$\phi(f) \approx c_o + c_1 f + \frac{c_2 f^2}{2!} + \frac{c_3 f^3}{3!}$$

where

$$c_3 = \frac{6\pi Z X^2}{c \, f_o^2 (1 - X^2)^{5/2}} \tag{74}$$

and

$$X = f_c/f_o$$

The quadratic analysis for small phase errors, given in Section III, is easily extended to include cubic phase errors. The output time signal $y(t)$ is found to be

$$y(t) = S_o(t) + S_3(t)$$

where

$$S_o(t) = \frac{\sin \pi W t}{\pi W t} \tag{75}$$

and

$$S_3(t) = \frac{1}{[2\pi]^3} \, j \, \frac{c_3}{3!} \, \frac{d^3 S_o(t)}{dt^3}$$

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Evaluating the third derivative at $t = \ell/W$ gives

$$S_3\left(\tfrac{\ell}{W}\right) = \frac{jc_3}{(2\pi)^3\, 3!} \; \frac{W^3(-1)^\ell}{\ell}\left[\frac{6}{\ell^2} - \pi^2\right] \tag{76}$$

Only the term of $S_3(\ell/W)$ which drops off as $1/\ell^3$ is truly a cubic distortion. The other portion which drops off as $1/\ell$ corresponds to a pure envelope delay or a linear phase distortion. The amount of the pure envelope delay is proportional to the slope of a straight line passing through $\phi(f)$ at both edges of the band (see Figure 20). That is

$$\tau_c = \frac{1}{W}\left[\phi\left[f_o + \frac{W}{2}\right] - \phi\left[f_o - \frac{W}{2}\right]\right] \tag{77}$$

Thus, the cubic distortion can be written

$$S_3\left[\tfrac{\ell}{W}\right] = S_3'\left[\tfrac{\ell}{W}\right] + S_1\left[\tfrac{\ell}{W}\right] \tag{78}$$

where

$$S_3'\left[\tfrac{\ell}{W}\right] = S_3\left[\tfrac{\ell}{W}\right] + \frac{jc_3 W^3 (-1)^\ell}{(2\pi)^3\, 3!} \cdot \frac{\pi^2}{\ell}$$

The following table gives the magnitude of $S_3'(\ell/W)$ in db per meter for various cut-off frequencies.

Table V

Magnitude of Cubic Phase Fourier Coefficients

ℓ \ $\dfrac{f_c}{f_o}$	.5	.7	.8	.9
1	-57 db	-50 db	-40 db	-24 db
3	-76 db	-79 db	-68 db	-52 db
5	-99 db	-92 db	-82 db	-66 db

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$\phi_{(f)}$

τ = ENVELOPE DELAY
= − SLOPE
= AVE $\left[\tau_g(f)\right]$

$-\dfrac{w}{2}$

$+\dfrac{w}{2}$

$f - f_o$
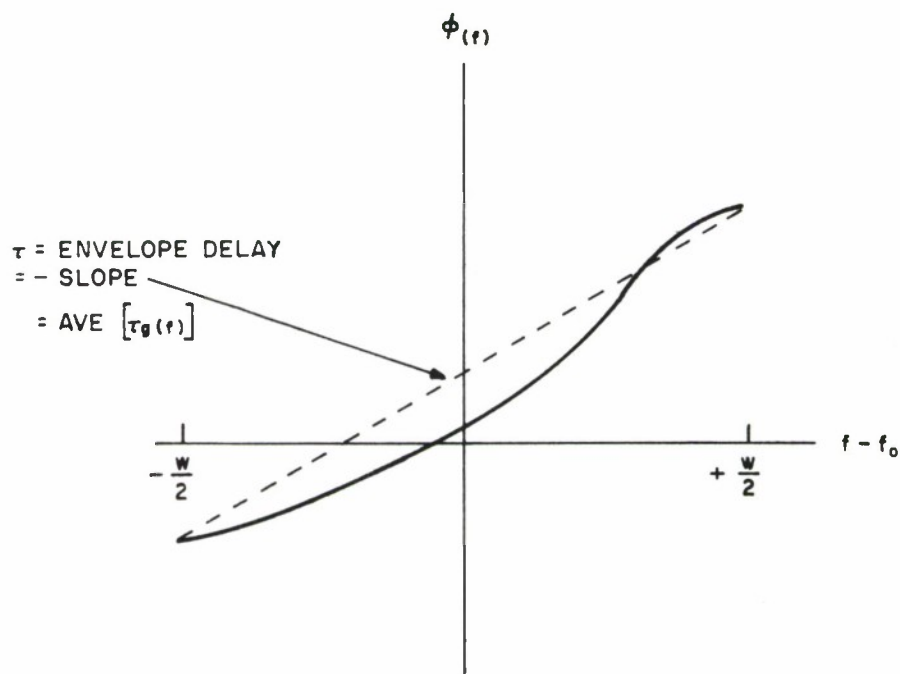
$$\tau_g(f) = -\frac{1}{2\pi}\ \frac{d\,\phi(f)}{d\,f}$$

Figure 20. Envelope Delay Definition

To assess the relative importance of third-order terms for dispersive waveguide it must be compared with the effect of the quadratic term. From Equation 69, the max quadratic phase in a band W is found to be

$$\Phi = \frac{\pi W^2 Z X^2}{4 f_o c[1 - X^2]^{3/2}} \qquad (79)$$

Sidelobes due to quadratic distortion (see Equation 38) may be expressed

$$\lambda_\ell = j \frac{2\Phi (-1)^\ell}{\pi^2 \ell^2} \qquad (80)$$

Comparing the first sidelobe due to the cubic distortion, $S_3'(\frac{1}{W})$, with the first sidelobe due to quadratic distortion λ_1,

$$\frac{\lambda_1}{S_3'(\frac{1}{W})} = (\frac{W}{f_o}) \frac{3}{2\pi(1 - X^2)} \qquad (81)$$

Thus, even with $X = .80$ (i.e., $f_c = .80 f_o$) the effect of the cubic term is much smaller.

Although the cubic phase distortion is negligible, relative to the quadratic phase distortion for waveguide, there may be situations for which this is not the case. For example, if the quadratic phase is accounted for by the signal processor the cubic term would limit performance.

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SECTION VII

SUMMARY

The paper has emphasized a straightforward method of computing a distorted matched filter output for band-limited signals. This method is certainly not new but has somehow been overlooked with the advent of complicated phase-coded signals. By observing that phase coding is cancelled in the matched filter receiver, the required calculation can be written as a simple Fourier transform.

The principal of paired echoes is also well known. If there is a sinusoidal phase error in the band of interest then a series of error sidelobes will be introduced. If the phase error is less than one radian only a pair of error sidelobes are generated which are amplitude symetric about the mainlobe. If there are N cycles of sinusoidal phase error then the paired echoes are N resolution cells away from the mainlobe.

By Fourier theory it is possible to expand an arbitrary phase error function into a Fourier series (a sum of sines and cosines). Since the matched filters are assumed linear, the distorted output signal is a sum of paired echoes, the kth echo being associated with the kth Fourier coefficient.

If a rectangular envelope (in the frequency domain) unweighted signal is considered, the sample of its output at time k/W is the kth paired echo.

Thus, for an unweighted, distorted signal the output is directly related to the Fourier coefficients of the distortion function.

A weighted signal, on the other hand, cannot be expressed in this way. The distorted weighted output time signal is represented by a sum of paired echoes, each a delayed replica of the undistorted weighted signal; however, the weighted echoes are not orthogonal. Thus, samples of the output signal are a more complicated function of the Fourier coefficients of the distortion function. It was found that the kth sample was a convolution of the Fourier coefficients of the distortion function and the Fourier coefficients of the weighting function.

Therefore, to calculate the distortion of some filter, not only must the individual Fourier coefficients of the distorting filter be found, but their structure and relationship must be examined.

The quadratic phase dispersion frequently encountered in practice is analyzed with the Fourier theory. It was found that the error coefficients could be quite large for modest quadratic distortion, but they had a very special character. The coefficients alternated in sign, dropped off as the square of their number, and were 90° out of phase with respect to the mainlobe. This information is sufficient to predict that most of the sidelobes due to quadratic phase dispersion would be removed by sidelobe weighting. The reduction in the effect of the Fourier coefficients due to weighting is not expected to occur near the origin. In fact, an argument using

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stationary phase to approximate the Fourier integral predicts that the mainlobe will widen. Numerical results did in fact verify that the error sidelobes were not significant but that the mainlobe width grows considerably with increasing quadratic phase. This result is easily extended to include small amplitude and phase errors which are represented as a power series.

$$\phi(f) = \sum_{k=2}^{\infty} a_k \ (f - f_o)^k$$

This class of error will have adjacent Fourier coefficients which will interact upon sidelobe weighting resulting in a reduction of their effect. As noted in the text, these coefficients tend to broaden the mainlobe.

These ideas can be summarized by noting that phase (or amplitude) errors which are easily represented by a few terms of a Fourier series, i.e., basically sinusoidal, will generally degrade the sidelobe level, while phase errors which are easily represented by a few terms of a power series will generally widen the mainlobe. Most distortions fall into one of these categories; therefore, the application of these results should be relatively straightforward. To show this, the outline of different types of distortion are presented in the text. These include frequency-dependent scattering and time dilation.

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13. ABSTRACT

The degradation of signal processor performance in terms of loss of resolution (i. e., increased main lobe width and increased sidelobes) is analyzed and calculated for dispersive media. The analysis gives insight into the distortion mechanisms and compares well with straightforward and precise numerical calculations. The distortion of wideband signals is described for the following situations: Ionospheric dispersion (above plasma frequency), waveguide operated near cutoff, time dilation, and frequency dependent scattering cross section.

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